Rigidity and Vanishing Theorems in K-Theory I

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Abstract. In this paper, we first establish a K-theory version of the equivariant family index theorem for a circle action, then use it to prove several rigidity and vanishing theorems on the equivariant K-theory level.

0 Introduction. In [**W**], Witten considered the indices of elliptic operators on the free loop space $\mathcal{L}M$ of a spin manifold M. In particular the index of the formal signature operator on the loop space is exactly the elliptic genus of Landweber-Stong [**LS**]. Motivated by physics, Witten made the conjectures about the rigidity of these elliptic operators which says that their S^1-equivariant indices on M are independent of $g \in S^1$. See [**L**] for the history of the subject.

These conjectures were proved by Taubes [**T**], Bott-Taubes [**BT**], and by Hirzebruch [**H**] and Krichever [**K**]. Many aspects of mathematics are involved in their proofs. Taubes used analysis of Fredholm operators and Witten's interpretation of the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula; Krichever used cobordism; Bott and Taubes and Hirzebruch used the Atiyah-Bott-Segal-Singer Lefschetz fixed point formula. In [**Liu1**], it was observed that these rigidity theorems are consequences of their modular invariance. This allowed Liu ([**Liu1, 2**]) to give a simple and unified proof, as well as various further generalizations, of the above conjectures of Witten. In particular, several new vanishing theorems were found in [**Liu1, 2**].

In many situations in geometry, it is rather natural and necessary to generalize the above rigidity and vanishing theorems to the family case. For example, to use elliptic operators to study the fundamental groups of a manifold, one can reduce the question to a problem involving a family of elliptic operators. In [**LiuMa1**] and [**LiuMa2**], Liu and Ma proved several family rigidity and vanishing theorems. Such theorems contain higher level vanishing terms which will be useful for our understanding of the relationship between group actions and fundamental groups.

To be more precise, let M, B be two compact smooth manifolds, and $\pi : M \to B$ be a smooth fibration with compact fibre X. Assume that a compact Lie group G acts fiberwise on M, that is, the action preserves each fiber of π. Let P be a family of G-equivariant elliptic operators along the fiber X. Then the family index of P, in the sense of Atiyah and Singer [**AS**],

$$(0.1) \qquad \mathrm{Ind}(P) = \mathrm{Ker}P - \mathrm{Coker}P \in K_G(B),$$

is well-defined. Note that $\mathrm{Ind}(P)$ is a virtual G-representation. Let $\widehat{G}$ denote the space of all complex irreducible representations of G. By [**S**, Proposition 2.2], we have

$$(0.2) \qquad \mathrm{Ind}(P) = \oplus_{V \in \widehat{G}}\mathrm{Hom}_G(V, \mathrm{Ind}(P)) \otimes V$$

with $\mathrm{Hom}_G(V, \mathrm{Ind}(P)) \in K(B)$. We denote by $(\mathrm{Ind}(P))^G \in K(B)$ the G-invariant part of $\mathrm{Ind}(P)$.

A family of elliptic operator P is said to be *rigid on the equivariant Chern character level* with respect to this G-action, if the equivariant Chern character $\mathrm{ch}_g(\mathrm{Ind}(P)) \in H^*(B)$ is independent of $g \in G$. If $\mathrm{ch}_g(\mathrm{Ind}(P))$ is identically zero for any g, then we say P has *vanishing property on the equivariant Chern character level*. More generally, we say that P is *rigid on the equivariant K-theory level*, if $\mathrm{Ind}(P) = (\mathrm{Ind}(P))^G$. If this index is identically zero in $K_G(B)$, then we say that P has *vanishing property on the equivariant K-theory level*. To study rigidity and vanishing properties of $\mathrm{Ind}(P)$, it is clear that we only need to restrict to the case where $G = S^1$. From now on we make the assumption that $G = S^1$.

Note that the K-theory level rigidity and vanishing properties are more subtle than those on the Chern character level. The reason is that, by taking the Chern character, some torsion elements involved in the index bundle might be killed. Such torsion elements may appear in the study of fundamental groups, which we hope to pursue in the future.

In [**LiuMa1**], Liu and Ma proved that the elliptic genera are actually rigid on the equivariant Chern character level. Several vanishing theorems for equivariant Chern characters of these index bundles are also proved in [**LiuMa1**]. Motivated by the family rigidity theorem of [**LiuMa1**, Theorem 2.1], it is rather natural to expect that the elliptic genera have rigidity and vanishing properties on the equivariant K-theory level. The purpose of the present paper is to show that this is indeed the case.

To achieve our purpose, we first establish a K-theory version of the equivariant family index theorem [**LiuMa1**, Theorem 1.1] for S^1-action. However, we are not able to derive this formula directly by applying the localization formula in the equivariant K-theory as in [**ASe**], as the localizing process will also kill the torsion element in $K_{S^1}(B)$. Here instead, we combine the analysis of Wu-Zhang [**WuZ**, §3], which in turn relies on the technique of Bismut-Lebeau [**BL**], with a deformation trick of Zhang [**Z**, §2], which allows us to avoid the small eigenvalues problem, to establish such a formula.

To prove the main results of this paper, which are stated in Section 2.1, we will introduce some shift operators on certain vector bundles over the fixed point set of the circle action, and compare the index bundles after the shift operation. Then we get a recursive relation of these index bundles which will in turn lead us to the final result. This part is essentially a reformulation of the basic ideas of Taubes [**T**]. Our main observation here is that we can directly do the shift operations on the fixed point set by applying the K-theory version of the equivariant family index theorem. In this way, we avoid the construction of the Dirac operator on the normal bundle in the loop space, as well as the associated analysis on the Fredholm properties of these operators (cf. [**T**]). Consequently, some of the shifting operations we will construct are not the same as that in [**T**]. This simplifies the computation significantly. In fact, in some sense the proof we will present may be considered as a rather subtle K-theory version of the proof of Bott-Taubes [**BT**].

In a subsequent paper we will use the method in this paper to prove several rigidity and vanishing theorems on the equivariant K-theory level for Spin^c-manifolds, and also for almost complex manifolds.

This paper is organized as follows. In Section 1, we prove a K-Theory version of the equivariant family index theorem for circle action. In Section 2, we prove the rigidity and

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vanishing theorems of elliptic genera on the equivariant K-theory level. The proofs of the main results in Section 2 are based on two intermediate results which will be proved in Sections 3 and 4 respectively.

Some of the results of this paper have been announced in [**LiuMaZ**].

1 A K-theory version of the equivariant family index theorem

In this section, we will prove a K-theory version of the equivariant family index theorem [**LiuMa1**, Theorem 1.1] for S^1-action, which will play a crucial role in the following sections. As an immediate consequence, we obtain a K-theory version of the famous $\widehat{A}$-vanishing theorem of Atiyah and Hirzebruch [**AH**] for compact connected spin manifolds with non-trivial S^1-actions. As was pointed out in the introduction, different from the case of the usual index, we can not get the K-theory index formula directly by applying the localization formula in equivariant K-theory as in [**ASe**], as the localizing process will kill the torsion element in $K_{S^1}(B)$. So the formula we will derive is more precise and subtle for the S^1-action case.

This section is organized as follows: In Section 1.1, we state a K-theory version of the equivariant family index theorem for S^1-action on a family of spin manifolds. In Section 1.2, we prove Theorem 1.1 by applying the techniques of [**BL**, Sections 8, 9], [**WuZ**, Section 3] and a deformation trick in [**Z**]. In Section 1.3, we generalize Theorem 1.1 to somewhat more general situation. In particular, we obtain a K-theory version of the equivariant family index theorem for S^1-action on a family of Spin^c Dirac operators.

1.1 A K-theory version of the equivariant family index theorem

Let M, B be two compact manifolds, and $\pi : M \to B$ a smooth fibration with compact fibre X such that $\dim X = 2l$. Let TX denote the relative tangent bundle. Let W be a complex vector bundle over M and h^W an Hermitian metric on W.

Let h^{TX} be a Riemannian metric on TX and ∇^{TX} be the corresponding Levi-Civita connection on TX along the fibre X. Then the Clifford bundle $C(TX)$ is the bundle of Clifford algebras over M whose fibre at $x \in M$ is the Clifford algebra $C(T_xX)$ of (TX, h^{TX}).

We assume that the bundle TX is spin over M. Let $S(TX) = S^+(TX) \oplus S^-(TX)$ be the spinor bundle of TX. We denote by $c(\cdot)$ the Clifford action of $C(TX)$ on $S(TX)$. Let $\{e_i\}$ be an oriented orthonormal basis of (TX, h^{TX}), let $\{e^i\}$ be its dual basis. Let

$$(1.1) \qquad \tau = i^l c(e_1) \cdots c(e_{2l})$$

be the involution of $S(TX)$. Then $\tau|_{S^\pm(TX)} = \pm 1$.

Let $\nabla^{S(TX)}$ be the Hermitian connection on $S(TX)$ induced by ∇^{TX}. Let ∇^W be an Hermitian connection on (W, h^W). Let $\nabla^{S(TX)\otimes W}$ be the connection on $S(TX)\otimes W$ along the fibre X:

$$(1.2) \qquad \nabla^{S(TX)\otimes W} = \nabla^{S(TX)} \otimes 1 + 1 \otimes \nabla^W.$$

For $b \in B$, we denote by $E_b, E_{\pm,b}$ the set of $\mathcal{C}^\infty$-sections of $S(TX)\otimes W$, $S^\pm(TX)\otimes W$ over the fiber X_b. We regard E_b as the fibre of a smooth $\mathbf{Z}_2$-graded infinite dimensional vector bundle over B. Smooth sections of E over B will be identified to smooth sections of $S(TX) \otimes W$ over M.

Definition 1.1 *Define the twisted Dirac operator to be*

$$(1.3) \qquad D^X = \sum_i c(e_i)\nabla^{S(TX)\otimes W}_{e_i}.$$

Then D^X is a family Dirac operator which acts fiberwise on the fibers of π. For $b \in B$, D_b^X denote the restriction of D^X to the fibre E_b. D^X interchanges E_+ and E_-. Let $D_\pm^X$ be the restrictions of D^X to $E_\pm$.

Now we assume that S^1 acts fiberwise on M. We will consider that S^1 acts as identity on B. Without loss of generality we can assume that S^1 acts on (TX, h^{TX}) isometrically. We also assume that the action of S^1 lifts to $S(TX)$ and W, and commutes with ∇^W.

From [**LiuMa1**, Proposition 1.1], we know that the difference bundle over B,

$$(1.4) \qquad \operatorname{Ind}(D^X) = \operatorname{Ker}D_{+,b}^X - \operatorname{Ker}D_{-,b}^X,$$

is well-defined in the equivariant K-group $K_{S^1}(B)$.

We denote $[n]$ ($n \in \mathbf{Z}$) the one dimensional complex vector space on which S^1 acts as multiplication by g^n for a generator $g \in S^1$. By [**S**, Proposition 2.2], we know that there exist $A \in \mathbf{N}$, $R_n \in K(B)$ ($|n| \leq A$) such that

$$(1.5) \qquad \operatorname{Ind}(D^X) = \bigoplus_{|n| \leq A} R_n \otimes [n],$$

as an identification of S^1-difference bundles.

For $n \in \mathbf{Z}$, let $E_{+,b}^n$, $E_{-,b}^n$ be the subspaces of $E_{+,b}$, $E_{-,b}$ where S^1 acts as multiplication by g^n for $g \in S^1$. Then we can consider $E_b^n = E_{+,b}^n \oplus E_{-,b}^n$ as the fibres of a smooth $\mathbf{Z}_2$-graded infinite dimensional vector bundle over B. Smooth sections of E^n over B will be identified to smooth sections of $S(TX) \otimes W$ over M on which S^1 acts as multiplication by g^n for $g \in S^1$. Then the difference bundle over B,

$$(1.6) \qquad \operatorname{Ind}(D^X, n) = \operatorname{Ker}D_{+,b}^X|_{E_{+,b}^n} - \operatorname{Ker}D_{-,b}^X|_{E_{-,b}^n},$$

is well-defined in the K-group $K(B)$.

By the construction of the index bundle $\operatorname{Ind}(D^X)$ (cf. [**LiuMa1**, Proposition 1.2]), one knows well that

$$(1.7) \qquad \begin{aligned} \operatorname{Ind}(D^X, n) \ &= R_n, \quad \text{for} \quad |n| \leq A, \\ &= 0, \quad \text{for} \quad |n| > A. \end{aligned}$$

Let $F = \{F_\alpha\}$ be the fixed point set of the circle action on M. Then $\pi : F_\alpha \to B$ (resp. $\pi : F \to B$) is a smooth fibration with fibre Y_α (resp. Y), and Y is a totally geodesic compact submanifold in each fiber X. Let $\widetilde{\pi} : N \to F$ denote the normal bundle to F in M. Then $N = TX/TY$. We identify N as the orthogonal complement of TY in $TX_{|F}$. Let h^{TY}, h^N be the corresponding metrics on TY, N induced by h^{TX}. Then, we have the following S^1-equivariant decomposition of TX when restricted to F,

$$TX_{|F} = N_{m_1} \oplus \cdots \oplus N_{m_l} \oplus TY,$$

where each N_γ is a complex vector bundle such that $g \in S^1$ acts on it by g^γ. To simplify the notation, we will simply write that

$$(1.8) \qquad TX_{|F} = \oplus_{v \neq 0} N_v \oplus TY,$$

where N_v is a complex vector bundle such that $g \in S^1$ acts on it by g^v with $v \in \mathbf{Z}^*$. Clearly, $N = \oplus_{v \neq 0} N_v$. We will denote N as a complex vector bundle, and $N_{\mathbf{R}}$ the underlying real vector bundle, of N.

Since $N_{\mathbf{R}}$ is naturally oriented by the complex structure on N, TY is naturally oriented by the orientations of TX and N. Similarly, let

$$(1.9) \qquad\qquad W_{|F} = \oplus_v W_v$$

be the S^1-equivariant decomposition of the restriction of W over F. Here W_v ($v \in \mathbf{Z}$) is a complex vector bundle over F on which $g \in S^1$ acts by g^v.

Let $C(N_{\mathbf{R}})$ be the Clifford algebra bundle of $(N_{\mathbf{R}}, h^N)$. Then $\Lambda(\overline{N}^*)$ is a $C(N_{\mathbf{R}})$-Clifford module. Namely, if $U \in N$, let $U' \in \overline{N}^*$ correspond to U by the metric h^{TX}. If $U \in N$, we write

$$(1.10) \qquad\qquad c(U) = \sqrt{2} U' \wedge, \quad c(\overline{U}) = -\sqrt{2} i_{\overline{U}}.$$

On F, let $\omega_2(TY)$, $\omega_2(N)$ be the second Stiefel-Whitney classes of TY, N respectively. Let $\det N_v$, $\det N = \otimes_v \det N_v$ be the determinant line bundles of N_v, N over F. Let $c_1(\det N)$ be the first Chern class of $\det N$. As TX is spin, one gets the following identity in $H^2(F, \mathbf{Z}_2)$,

$$(1.11) \qquad\qquad \omega_2(TY) = \omega_2(N) = c_1(\det N) \mod (2).$$

As being explained in [**LaM**, Appendix D, pp. 397], we can construct a Spin^c -principal bundle and a complex spinor bundle $S(TY, (\det N)^{-1})$ over F which locally may be thought of as

$$S\left(TY, (\det N)^{-1}\right) = S_0(TY) \otimes (\det N)^{-1/2},$$
$$\Lambda(\overline{N}^*) = S_0(N_{\mathbf{R}}) \otimes (\det N)^{1/2},$$

where $S_0(TY)$, $S_0(N_{\mathbf{R}})$ are the fundamental spinor bundles for the (possibly nonexistent) spin structure on TY, $N_{\mathbf{R}}$, and where $(\det N)^{-1/2}$ is the (possibly nonexistent) square root of $(\det N)^{-1}$.

Since Y is totally geodesic in X, the connection $\nabla^{TX}|_F$ also preserves the decomposition (1.8) of TX over F. Let ∇^{TY}, ∇^N, ∇^{N_v} be the corresponding induced connections on TY, N and N_v, Let $\nabla^{\Lambda(\overline{N}^*)}$ be the connection on $\Lambda(\overline{N}^*)$ induced by ∇^N.

Let $\nabla^{(\det N)^{-1}}$ be the connection on $(\det N)^{-1}$ induced by ∇^N. The connections $\nabla^{(\det N)^{-1}}$ and ∇^{TY} induce an Hermitian connection $\nabla^{S(TY, (\det N)^{-1})}$ on $S(TY, (\det N)^{-1})$. In fact, locally, both $S_0(TY)$ and $(\det N)^{-1/2}$ exist and carry a canonical connection induced by ∇^{TY}, $\nabla^{(\det N)^{-1}}$. We give the bundle $S(TY, (\det N)^{-1})$ the tensor product connection. It is standard that this connection is well defined globally (cf. [**LaM**, Appendix D]).

Let $(e_1, \cdots, e_{2l'})$, $(e_{2l'+1}, \cdots, e_{2l})$ be the corresponding oriented orthonormal basis of TY and $N_{\mathbf{R}}$. Then $(e_1, \cdots, e_{2l})$ is an oriented orthonormal basis of TX. The $\mathbf{Z}_2$-gradings on $S(TY, (\det N)^{-1})$, $\Lambda(\overline{N}^*)$ are defined by the involutions $i^{l'} c(e_1) \cdots c(e_{2l'})$, and $\tau^N = i^{l-l'} c(e_{2l'+1}) \cdots c(e_{2l})$ respectively. Also note that under this involution, $\Lambda^{\text{even}}(\overline{N}^*) = (\Lambda(\overline{N}^*))^+$, $\Lambda^{\text{odd}}(\overline{N}^*) = (\Lambda(\overline{N}^*))^-$.

From the above discussion, we see that there is a natural isomorphism between $\mathbf{Z}_2$-graded $C(TX)$-Clifford modules over F,

$$(1.12) \qquad S\left(TY, (\det N)^{-1}\right) \widehat{\otimes} \Lambda\left(\overline{N}^*\right) \simeq S(TX)|_F.$$

Here we denote the $\mathbf{Z}_2$-graded tensor product by $\widehat{\otimes}$ (cf. [**LaM**, pp. 11]). Furthermore, since ∇^N is S^1-invariant, one deduces easily that

$$(1.13) \qquad \nabla^{S(TX)}|_F = \nabla^{S(TY,(\det N)^{-1})} \widehat{\otimes} 1 + 1 \widehat{\otimes} \nabla^{\Lambda(\overline{N}^*)}.$$

Let V be an Hermitian vector bundle over F. Let ∇^V be an Hermitian connection on V. From now on, we will also write $D^Y \otimes V$ the family twisted Dirac operator on $S(TY, (\det N)^{-1}) \otimes V$ on F, and $D^{Y_\alpha} \otimes V$ its restriction on F_α. Namely, let $\nabla^{S(TY,(\det N)^{-1})\otimes V}$ be the tensor product connection on $S(TY, (\det N)^{-1}) \otimes V$ induced by $\nabla^{S(TY,(\det N)^{-1})}$ and ∇^V. Then

$$(1.14) \qquad D^Y \otimes V = \sum_{i=1}^{2l'} c(e_i) \nabla_{e_i}^{S(TY,(\det N)^{-1})\otimes V}.$$

We use the notation

$$\mathrm{Sym}_q(V) = \sum_{n=0}^{+\infty} q^n \mathrm{Sym}^n(V) \in K(F)[[q]],$$
$$\Lambda_q(V) = \sum_{n=0}^{+\infty} q^n \Lambda^n(V) \in K(F)[[q]]$$

for the symmetric and exterior power operations in $K(F)[[q]]$ respectively.

Let us intorduce the notations:

$$(1.15) \qquad \begin{aligned} R(q) &= q^{\frac{1}{2}\sum_v |v|\dim N_v} \otimes_{v>0} \left(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\right) \\ &\quad \otimes_{v<0} \mathrm{Sym}_{q^{-v}}\left(\overline{N}_v\right) \otimes_v q^v W_v = \oplus_n R_n q^n, \\ R'(q) &= q^{-\frac{1}{2}\sum_v |v|\dim N_v} \otimes_{v>0} \mathrm{Sym}_{q^{-v}}\left(\overline{N}_v\right) \\ &\quad \otimes_{v<0} \left(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\right) \otimes_v q^v W_v = \oplus_n R'_n q^n. \end{aligned}$$

Note that by [**AH**], one knows that, as TX is spin,

$$(1.16) \qquad \sum_v v\dim N_v = 0 \mod (2).$$

We can now state the main result of this section as follows.

Theorem 1.1 *For $n \in \mathbf{Z}$, we have the following identity in $K(B)$,*

$$(1.17) \qquad \begin{aligned} \mathrm{Ind}\left(D^X, n\right) &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}\left(D^{Y_\alpha} \otimes R_n\right) \\ &= \sum_\alpha (-1)^{\Sigma_{v<0} \dim N_v} \mathrm{Ind}\left(D^{Y_\alpha} \otimes R'_n\right). \end{aligned}$$

If we take $W = \mathbf{C}$ the trivial line bundle over M, then the operator $D^X \otimes \mathbf{C}$ is exactly the canonical Dirac operator D^X. The following consequence generalizes the famous $\widehat{A}$-vanishing theorem of Atiyah and Hirzebruch [**AH**] for compact connected spin manifolds with non-trivial S^1-actions to the family case.

Corollary 1.1 *If M is connected, and the S^1 action is nontrivial, then for the family of the canonical Dirac operators D^X along the fibre X, one has*

$$(1.18) \qquad\qquad \mathrm{Ind}\left(D^X\right) = 0 \quad \text{in} \quad K_{S^1}(B).$$

Proof: If the S^1 action is locally free, then by Theorem 1.1, we get directly (1.18). Otherwise, since on each F_α,

$$(1.19) \qquad\qquad \sum_v |v| \dim N_v > 0,$$

one deduces easily from Theorem 1.1 that

$$(1.20) \qquad\qquad \mathrm{Ind}\left(D^X, n\right) = 0 \quad \text{in} \quad K(B)$$

for any $n \in \mathbf{Z}$, from which (1.18) follows. ∎

1.2 Proof of Theorem 1.1

In this subsection, we prove Theorem 1.1. The proof, which is contained here for completeness of the present paper, is modeled on [**WuZ**, Section 3] which in turn relies on the paper of Bismut and Lebeau [**BL**].

This subsection is organized as follows. In Section 1.2.1, we recall a result from [**WuZ**, Proposition 3.2] concerning the Witten deformation on flat space. In Section 1.2.2, we establish a Taylor expansion of $D^X + \sqrt{-1}Tc(H)$ near the fixed point set F, where H is the Killing vector field on M generated by the circle action. In Section 1.2.3, by using the techniques of [**WuZ**, Section 3] and [**BL**, Section 9], we establish various estimates for certain operators induced from $D^X + \sqrt{-1}Tc(H)$. In Section 1.2.4, we prove Theorem 1.1 by using a deformation trick in [**Z**].

1.2.1 Witten's deformation on flat spaces

Let H be the canonical basis of $\mathrm{Lie}(S^1) = \mathbf{R}$, i.e., for $t \in \mathbf{R}$, $\exp(tH) = e^{2\pi i t} \in S^1$. Let W be a complex vector space of dimension n with an Hermitian form. Let ρ be a unitary representation of the circle group S^1 on W such that all the weights are nonzero. Suppose $W^\pm$ are the subspaces of W corresponding to positive and negative weights respectively, with $\dim_{\mathbf{C}} W^- = \nu$, $\dim_{\mathbf{C}} W^+ = n - \nu$. Let $z = \{z^i\}$ be the complex linear coordinates on W such that the Hermitian structure on W takes the standard form and ρ is diagonal with weights $\lambda_i \in \mathbf{Z} \backslash \{0\}$ $(1 \le i \le n)$, and $\lambda_i < 0$ for $i \le \nu$. The Lie algebra action is given by the vector field $H = 2\pi\sqrt{-1}\Sigma_{i=1}^n \lambda_i(z^i \frac{\partial}{\partial z^i} - \overline{z}^i \frac{\partial}{\partial \overline{z}^i})$ on W. We write $K^\pm(W) = \mathrm{Sym}((W^\pm)^*) \otimes \mathrm{Sym}(W^\mp) \otimes \det(W^\mp)$. Let E be a finite dimensional complex vector space with an Hermitian form and suppose E carries an unitary representation of S^1.

Let $\overline{\partial}$ be the twisted Dolbeault operator acting on $\Omega^{0,*}(W, E) = \Gamma(\Lambda(\overline{W}^*) \otimes E)$, and $\overline{\partial}^*$ its formal adjoint. Let $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$. Let $c(H)$ be the Clifford action of H on $\Lambda(\overline{W}^*)$ defined as in (1.10). Let L_H be the Lie derivative along H acting on $\Omega^{0,*}(W, E)$.

The following result was proved in [**WuZ**, Proposition 3.2].

Proposition 1.1 *1. A basis of the space of L^2-solutions of $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) on the space of $\mathcal{C}^\infty$ sections of $\Lambda(\overline{W}^*)$ is given by*

$$(1.21) \qquad \Pi_{i=1}^\nu z_i^{k_i} \Pi_{i=\nu+1}^n \overline{z}_i^{k_i} e^{-\Sigma_{i=1}^n \pi |\lambda_i||z_i|^2} d\overline{z}_{\nu+1} \cdots d\overline{z}_n \quad (k_i \in \mathbf{N})$$

with weight $\Sigma_{i=1}^\nu k_i |\lambda_i| + \Sigma_{i=\nu+1}^n (k_i + 1)|\lambda_i|$ (resp.

$$(1.22) \qquad \Pi_{i=1}^\nu \overline{z}_i^{k_i} \Pi_{i=\nu+1}^n z_i^{k_i} e^{-\Sigma_{i=1}^n \pi |\lambda_i||z_i|^2} d\overline{z}_1 \cdots d\overline{z}_\nu \quad (k_i \in \mathbf{N})$$

with weight $-\Sigma_{i=\nu+1}^n k_i |\lambda_i| - \Sigma_{i=1}^\nu (k_i + 1)|\lambda_i|$).
So the space of L^2-solution of a given weight of $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) on the space of $\mathcal{C}^\infty$ sections of $\Lambda(\overline{W}^*) \otimes E$ is finite dimensional. The direct sum of these weight spaces is isomorphic to $K^-(W) \otimes E$ (resp. $K^+(W) \otimes E$) as represnetations of S^1.
2. When restricted to an eigenspace of L_H, the operator $D + \sqrt{-1}c(H)$ (resp. $D - \sqrt{-1}c(H)$) has discrete eigenvalues.

1.2.2 A Taylor expantion of certain deformed operators near the fixed-point set

In this subsection, we will use the notation of Section 1.1. We assume temporarily that B is a point, and Y is the fixed point set of the S^1 action on X.

Following [**BL**, Section 8e)] and [**WuZ**, Section 3.2], we now describe a coordinate system on X near Y.

For $\varepsilon > 0$, set $B_\varepsilon = \{Z \in N; |Z| < \varepsilon\}$. Since X and Y are compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the exponential map $(y, Z) \in N \to \exp_y^X(Z) \in X$ is a diffeomorphism from B_ε onto a tubular neighborhood U_ε of Y in X. From now on, we identify B_ε with U_ε and use the notation $x = (y, Z)$ instead of $x = \exp_y^X(Z)$. Finally, we identify $y \in Y$ with $(y, 0) \in N$.

Let h^{TY}, h^N be the corresponding metrics on TY and N induced by h^{TX}. Let dv_X, dv_Y and dv_N be the corresponding volume elements on (TX, h^{TX}), (TY, h^{TY}) and (N, h^N). Let $k(y, Z)$ $((y, Z) \in B_\varepsilon)$ be the smooth positive function defined by $dv_X(y, Z) = k(y, Z)dv_Y(y)dv_{N_y}(Z)$. Then $k(y) = 1$ and $\frac{\partial k}{\partial Z}(y) = 0$ for $y \in Y$; the latter follows from [**BL**, Proposition 8.9] and the fact that Y is totally geodesic in X.

For $x = (y, Z) \in U_{\varepsilon_0}$, we will identify $S(TX)_x$ with $S(TX)_y$ and W_x with W_y by the parallel transport with respect to the S^1-invariant connections $\nabla^{S(TX)}$ and ∇^W respectively, along the geodesic $t \to (y, tZ)$. The induced identification of $(S(TX) \otimes W)_x$ with $(S(TX) \otimes W)_y$ preserves the metric and the $\mathbf{Z}_2$-grading, and is moreover S^1-equivariant. Consequently, D^X can be considered as an operator acting on the sections of the bundle $\widetilde{\pi}^*(S(TX) \otimes W)|_Y)$ over B_{ε_0} commuting with the S^1 action.

For $\varepsilon > 0$, let $\mathbf{H}(B_\varepsilon)$ (resp. $\mathbf{H}(N)$) be the set of smooth sections of $\widetilde{\pi}^*(S(TX) \otimes W)|_Y$ on B_ε (resp. on the total bundle of N). If f, $g \in \mathbf{H}(N)$ have compact supports, we will write

$$(1.23) \qquad \langle f, g \rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_Y \left(\int_N \langle f, g \rangle (y, Z) dv_{N_y}(Z) \right) dv_Y(y).$$

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Then $k^{1/2}D^X k^{-1/2}$ is a (formal) self-adjoint operator on $\mathbf{H}(N)$.

The connection ∇^N on N induces a splitting $TN = N \oplus T^H N$, where $T^H N$ is the horizontal part of TN with respect to ∇^N. Moreover, since Y is totally geodesic, this splitting, when restricted to Y, is preserved by the connection ∇^{TX} on $TX|_Y$. Let $\widetilde{\nabla}$ be the connection on $(S(TX) \otimes W)|_Y$ induced by the restricion of $\nabla^{S(TX) \otimes W}$ to Y. We will still denote by $\widetilde{\nabla}$ the lift of the connection $\widetilde{\nabla}$ to $\widetilde{\pi}^*(S(TX) \otimes W)|_Y)$,.

We choose a local orthonormal basis of TX such that $e_1, \cdots, e_{2l'}$ form a basis of TY, and $e_{2l'+1}, \cdots, e_{2l}$, that of $N_{\mathbf{R}}$. Denote the horizontal lift of e_i $(1 \leq i \leq 2l')$ to $T^H N$ by e_i^H. As in [**BL**, Definition 8.16] and [**WuZ**, (3.15)], we define

$$(1.24) \qquad D^H = \sum_{i=1}^{2l'} c(e_i)\widetilde{\nabla}_{e_i^H}, \qquad D^N = \sum_{i=2l'+1}^{2l} c(e_i)\widetilde{\nabla}_{e_i}.$$

Clearly, D^N acts along the fibers of N. Let $\overline{\partial}^N$ be the $\overline{\partial}$-operator along the fibers of N, and let $\overline{\partial}^{N*}$ be its formal adjoint with respect to (1.23). By (1.13), it is easy to see that $D^N = \sqrt{2}(\overline{\partial}^N + \overline{\partial}^{N*})$. Both D^N and D^H are self-adjoint with respect to (1.23).

For $T > 0$, we define a scaling $f \in \mathbf{H}(B_{\varepsilon_0}) \to S_T f \in \mathbf{H}(B_{\varepsilon_0 \sqrt{T}})$ by

$$(1.25) \qquad S_T f(y, Z) = f\left(y, \frac{Z}{\sqrt{T}}\right), \qquad (y, Z) \in B_{\varepsilon_0 \sqrt{T}}.$$

For a first order differential operator

$$(1.26) \qquad Q_T = \sum_{i=1}^{2l'} a_T^i(y, Z)\widetilde{\nabla}_{e_i^H} + \sum_{i=2l'+1}^{2l} b_T^i(y, Z)\widetilde{\nabla}_{e_i} + c_T(y, Z)$$

acting on $\mathbf{H}(B_{\varepsilon_0 \sqrt{T}})$, where a_T^i, b_T^i and c_T are endomorphisms of $\widetilde{\pi}^*((S(TX) \otimes W)|_Y)$, we write

$$(1.27) \qquad Q_T = O\left(|Z|^2 \partial^N + |Z|\partial^H + |Z| + |Z|^p\right), \qquad (p \in \mathbf{N}),$$

if there is a constant $C > 0$ such that for any $T \geq 1$, $(y, Z) \in B_{\varepsilon_0 \sqrt{T}}$, we have

$$(1.28) \qquad \begin{aligned} &|a_T^i(y, Z)| \leq C|Z| \qquad (1 \leq i \leq 2l'), \\ &|b_T^i(y, Z)| \leq C|Z|^2 \qquad (2l' + 1 \leq i \leq 2l), \\ &|c_T(y, Z)| \leq C\left(|Z| + |Z|^p\right). \end{aligned}$$

Let J_H be the representation of $\mathrm{Lie}(S^1)$ on N. Then $Z \to J_H Z$ is a Killing vector field on N. We have the following analog of [**BL**, Theorem 8.18] and [**WuZ**, Proposition 3.3].

Proposition 1.2 *As* $T \to +\infty$,

$$(1.29) \qquad \begin{aligned} S_T k^{1/2}\left(D^X + \sqrt{-1}Tc(H)\right)k^{-1/2}S_T^{-1} &= \sqrt{T}\left(D^N + \sqrt{-1}c(J_H Z)\right) + D^H \\ &+ \tfrac{1}{\sqrt{T}}O\left(|Z|^2 \partial^N + |Z|\partial^H + |Z| + |Z|^3\right). \end{aligned}$$

Proof: Since Y is totally geodesic in X and the actions of S^1 on N and M commute with the exponential map, one can proceed as in the proof of [**WuZ**, Proposition 3.3] and [**BL**, Section 8] to get (1.29). ∎

By Proposition 1.1, the solution space of the operator $D^N + \sqrt{-1}c(J_H Z)$ along the fiber N_y ($y \in F$) is (the L^2-completion of) $K^-(N_y) \otimes W_y$. They form an infinite dimensional Hermitian holomorphic vector bundle $K^-(N) \otimes W|_F$ over Y, with the Hermitian connection ∇^Y induced from those on N and $W|_Y \to Y$.

Let $\mathbf{H}^0(Y)$ be the Hilbert space of square-integrable sections of $S(TY, (\det N)^{-1}) \otimes K^-(N) \otimes W|_Y$, and $\mathbf{H}^0(N)$, that of the bundle $\widetilde{\pi}^*((S(TX) \otimes W)|_Y)$, equipped with the corresponding Hermitian forms. By using (1.12), we define an embedding $\psi : \mathbf{H}^0(Y) \to \mathbf{H}^0(N)$ by

$$(1.30) \qquad \psi \colon \alpha \otimes \beta \in \mathbf{H}^0(Y) \to \widetilde{\pi}^*\alpha \wedge \tau(\beta) \in \mathbf{H}^0(N).$$

Here $\alpha \in \Gamma(Y, S(TY, (\det N)^{-1}))$, $\beta \in L^2(K^-(N) \otimes W|_Y)$ and τ is the isometry from $L^2(K^-(N) \otimes W|_Y)$ to $L^2(\widetilde{\pi}^*(\Lambda(\overline{N}^*) \otimes W|_Y))$ given by Proposition 1.1. Clearly, ψ is an isometry onto its image which we denote by $\mathbf{H}'^{,0}$. Let $p : \mathbf{H}^0(N) \to \mathbf{H}'^{,0}$ be the orthogonal projection. Then we have the following analog of [**BL**, Theorem 8.21] and [**WuZ**, Proposition 3.4], which can be proved in the same way as in [**BL**] and [**WuZ**].

Proposition 1.3 *The following identity for operators acting on* $\mathbf{H}^0(Y)$ *holds,*

$$(1.31) \qquad \psi^{-1} p\, D^H p\, \psi = D^Y \otimes R(1),$$

where $R(1)$ is defined in (1.15).

1.2.3 Estimates of the operators as $T \to +\infty$

We still assume temporarily that B is a point. For $p \geq 0$, let $\mathbf{H}^p(X)$, $\mathbf{H}^p(N)$ and $\mathbf{H}^p(Y)$ be the p-th Sobolev spaces of sections of the bundles $S(TX) \otimes W \to X$, $\widetilde{\pi}^*(S(TX) \otimes W)|_Y \to N$ and $S(TY, (\det N)^{-1}) \otimes K^-(N) \otimes W|_Y \to Y$ respectively. The group S^1 acts on all these spaces. For any $\xi \in \mathbf{Z}$, let $\mathbf{H}^p_\xi(X)$, $\mathbf{H}^p_\xi(N)$ and $\mathbf{H}^p_\xi(Y)$ be the corresponding subspaces of weight $\xi \in \mathbf{Z}$. Recall that the constant $\epsilon_0 > 0$ is defined in last subsection. We now take $\epsilon \in (0, \frac{\epsilon_0}{2}]$, which is small enough for each eigenvalue of L_H we will consider, but otherwise can be assumed to be fixed. Let $\rho \colon \mathbf{R} \to [0, 1)$ be a smooth function such that

$$(1.32) \qquad \rho(a) = \begin{cases} 1 & \text{if} \quad a \leq 1/2, \\ 0 & \text{if} \quad a \geq 3/4. \end{cases}$$

For $Z \in N$, set $\rho_\epsilon(Z) = \rho(\frac{|Z|}{\epsilon})$. Let $\alpha \in \Gamma(Y, S(TY, (\det N)^{-1}))$, $\beta \in L^2(K^-(N) \otimes W|_Y)$. We define a linear map $I_{T,\xi} : \mathbf{H}^p_\xi(Y) \to \mathbf{H}^p_\xi(N)$ by

$$(1.33)\; \alpha \otimes \beta \in \mathbf{H}^p_\xi(Y) \longmapsto I_{T,\xi}\sigma = \frac{\rho_\epsilon \|\beta\|_0}{\|\rho_\epsilon S_T^{-1}(\tau(\beta))\|_0} \widetilde{\pi}^*\alpha \wedge S_T^{-1}(\tau(\beta)) \in \mathbf{H}^p_\xi(N).$$

Let the image of $I_{T,\xi}$ from $\mathbf{H}_\xi^p(Y)$ be $\mathbf{H}_{T,\xi}^p(N) = I_{T,\xi}\mathbf{H}^p(Y) \subset \mathbf{H}_\xi^p(N)$. Denote the orthogonal complement of $\mathbf{H}_{T,\xi}^0(N)$ in $\mathbf{H}_\xi^0(N)$ by $\mathbf{H}_{T,\xi}^{0,\perp}(N)$, and let $\mathbf{H}_{T,\xi}^{p,\perp}(N) = \mathbf{H}_\xi^p(N) \cap \mathbf{H}_{T,\xi}^{0,\perp}(N)$. Let $p_{T,\xi}$ and $p_{T,\xi}^\perp$ be the orthogonal projections from $\mathbf{H}_\xi^0(N)$ onto $\mathbf{H}_{T,\xi}^0(N)$ and $\mathbf{H}_{T,\xi}^{0,\perp}(N)$ respectively.

Since the bundle $S(TX)\otimes W$ over U_{ϵ_0} is identified with $\widetilde{\pi}^*(S(TY,(\det N)^{-1})\otimes\Lambda(\overline{N}^*)\otimes W|_Y)$ over B_{ϵ_0}, we can consider $k^{-1/2}I_{T,\xi}\sigma$ as an element of $\mathbf{H}_\xi^p(X)$ for $\sigma \in \mathbf{H}_\xi^p(Y)$. Define the linear map $J_{T,\xi}$ by

$$(1.34) \qquad \sigma \in \mathbf{H}_\xi^p(Y) \longmapsto J_{T,\xi}\sigma = k^{-1/2}I_{T,\xi}\sigma \in \mathbf{H}_\xi^p(X).$$

Let $\mathbf{H}_{T,\xi}^p(X) = J_{T,\xi}\mathbf{H}^p(Y)$ be the image. Denote the orthogonal complement of $\mathbf{H}_{T,\xi}^0(X)$ in $\mathbf{H}_\xi^0(X)$ by $\mathbf{H}_{T,\xi}^{0,\perp}(X)$, and let $\mathbf{H}_{T,\xi}^{p,\perp}(X) = \mathbf{H}_\xi^p(X) \cap \mathbf{H}_{T,\xi}^{0,\perp}(X)$. Let $\bar{p}_{T,\xi}$ and $\bar{p}_{T,\xi}^\perp$ be the orthogonal projections from $\mathbf{H}_\xi^0(X)$ onto $\mathbf{H}_{T,\xi}^0(X)$ and $\mathbf{H}_{T,\xi}^{0,\perp}(X)$ respectively. It is clear that $\bar{p}_{T,\xi} = k^{-1/2}p_{T,\xi}k^{1/2}$.

For any (possibly unbounded) operator A on $\mathbf{H}_\xi^0(X)$, write

$$(1.35) \qquad A = \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix}$$

according to the decomposition $\mathbf{H}_\xi^0(X) = \mathbf{H}_{T,\xi}^0(X) \oplus \mathbf{H}_{T,\xi}^{0,\perp}(X)$, i.e., $A^{(1)} = \bar{p}_{T,\xi}A\bar{p}_{T,\xi}$, $A^{(2)} = \bar{p}_{T,\xi}A\bar{p}_{T,\xi}^\perp$, $A^{(3)} = \bar{p}_{T,\xi}^\perp A\bar{p}_{T,\xi}$ and $A^{(4)} = \bar{p}_{T,\xi}^\perp A\bar{p}_{T,\xi}^\perp$.

Let $D_T = D^X + \sqrt{-1}Tc(H)$, where now H denotes the Killing vector field on M generated by the circle action. Let $D_{T,\xi}$ and D_ξ^Y be the restrictions of the operators D_T and $D^Y \otimes R(1)$ on $\mathbf{H}_\xi^0(X)$ and $\mathbf{H}_\xi^0(Y)$ respectively.

Proposition 1.4 *1. As $T \to +\infty$,*

$$(1.36) \qquad J_{T,\xi}^{-1}D_{T,\xi}^{(1)}J_{T,\xi} = D_\xi^Y + O\left(\frac{1}{\sqrt{T}}\right),$$

where $O(\frac{1}{\sqrt{T}})$ denotes a first order differential operator whose coefficients are dominated by $\frac{C}{\sqrt{T}}$ $(C > 0)$.

2. For each $\xi \in \mathbf{Z}$, there exists $C > 0$ such that for any $T \geq 1$, $\sigma \in \mathbf{H}_{T,\xi}^{1,\perp}(X)$, $\sigma' \in \mathbf{H}_{T,\xi}^1(X)$, we have

$$(1.37) \qquad \begin{aligned} \|D_{T,\xi}^{(2)}\sigma\|_0 &\leq C\left(\frac{\|\sigma\|_1}{\sqrt{T}} + \|\sigma\|_0\right), \\ \|D_{T,\xi}^{(3)}\sigma'\|_0 &\leq C\left(\frac{\|\sigma'\|_1}{\sqrt{T}} + \|\sigma'\|_0\right). \end{aligned}$$

3. For each $\xi \in \mathbf{Z}$, there exist $\epsilon \in (0,\frac{\epsilon_0}{2}]$, $T_0 > 0$, $C > 0$ such that for any $T \geq T_0$, $\sigma \in \mathbf{H}_{T,\xi}^{1,\perp}(X)$, we have

$$(1.38) \qquad \|D_{T,\xi}^{(4)}\sigma\|_0 \geq C\left(\|\sigma\|_1 + \sqrt{T}\|\sigma\|_0\right).$$

Proof: Proposition 1.4 is the analogue of [**WuZ**, Proposition 3.5] and can be proved in the same way as in [**WuZ**, pp. 165-166], which in turn relies on [**BL**, Section 9]. ∎

1.2.4 Proof of Theorem 1.1

We now go back to the family case. The important observation is that the analysis in the above two subsections works well to the fiberwise (twisted) Dirac operators.

For any $u \in \mathbf{R}$, we write

$$(1.39) \qquad D_{T,\xi}(u) = D_{T,\xi}^{(1)} + D_{T,\xi}^{(4)} + u\left(D_{T,\xi}^{(2)} + D_{T,\xi}^{(3)}\right) : E \to E.$$

The following lemma plays a key role in our proof of Theorem 1.1.

Lemma 1.1 *There exists $T_1 > 0$ such that for any $u \in [0,1]$ and $T \geq T_1$, $D_{T,\xi}(u)$ is a continuous family of Fredholm operators over B.*

Proof: From Proposition 1.4, one deduces (cf. [**Z**, Lemma 2.2]) that there exist C_1, $C_2 > 0$ such that for $u \in [0,1]$, $s \in E$ and T large enough,

$$(1.40) \qquad ||D_{T,\xi}s - D_{T,\xi}(u)s||_0 \leq \frac{C_1}{\sqrt{T}}||D_{T,\xi}s||_0 + C_2||s||_0.$$

From (1.40) and the Fredholm property of $D_{T,\xi}$, one obtains the Fredholm property of $D_{T,\xi}(u)$ for sufficiently large T. ∎

Recall that the index bundle construction [**AS**] applies well to continuous families of Fredholm operators and that the homotopy invariance property for the index bundle still holds in this situation.

From Lemma 1.1, one then gets the following identity of index bundles, when T is large enough,

$$(1.41) \qquad \begin{aligned} \operatorname{Ind}\left(D_\xi^X\right) &= \operatorname{Ind}\left(D_{T,\xi}\right) = \operatorname{Ind}\left(D_{T,\xi}(0)\right) \\ &= \operatorname{Ind}\left(D_{T,\xi}^{(1)}\right) + \operatorname{Ind}\left(D_{T,\xi}^{(4)}\right) \quad \text{in} \quad K(B), \end{aligned}$$

where in the last line, $\operatorname{Ind}(D_{T,\xi}^{(1)})$ (resp. $\operatorname{Ind}(D_{T,\xi}^{(4)})$) is now regarded as a family of Fredholm operators mapping from $\mathbf{H}_{T,\xi}^0(X)$ (resp. $\mathbf{H}_{T,\xi}^{0,\perp}(X)$) to $\mathbf{H}_{T,\xi}^0(X)$ (resp. $\mathbf{H}_{T,\xi}^{0,\perp}(X)$).

On the other hand, by the third part of Proposition 1.4, one has obviously that

$$(1.42) \qquad \operatorname{Ind}\left(D_{T,\xi}^{(4)}\right) = 0 \quad \text{in} \quad K(B),$$

when T is large enough.

Let $D_{T,\xi,\alpha}^{(1)}$ be the restriction of $D_{T,\xi}^{(1)}$ on F_α. From (1.30), (1.36) and the definition of $J_{T,\xi}$, one deduces easily that when T is large enough, one has,

$$(1.43) \qquad \begin{aligned} \operatorname{Ind}\left(D_{T,\xi}^{(1)}\right) &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \operatorname{Ind}\left(J_{T,\xi}^{-1} D_{T,\xi,\alpha}^{(1)} J_{T,\xi}\right) \\ &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \operatorname{Ind}\left(D_\xi^{Y_\alpha}\right) \quad \text{in} \quad K(B). \end{aligned}$$

By (1.41), (1.42) and (1.43), one deduces the first equation of (1.17) easily.

To get the second equation of (1.17), we only need to apply the first equation of (1.17) to the S^1-action on M defined by the inverse of the original S^1-action on M.

The proof of Theorem 1.1 is complete. ∎

1.3   The Spin^c case

We will keep the notations in Sections 1.1 and 1.2. For future applications, in this subsection, we will extend Theorem 1.1 to Spin^c cases.

Let $\pi : M \to B$ be a fibration of compact manifolds with compact fibre X such that $\dim X = 2l$ and that S^1 acts fiberwise on M. Let h^{TX} be a metric on TX. We assume that TX is oriented. Let (W, h^W) be an Hermitian complex vector bundle over M.

Let V be a $2p$ dimensional oriented real vector bundle over M. Let L be a complex line bundle over M with the property that the vector bundle $U = TX \oplus V$ obeys $w_2(U) = c_1(L) \bmod (2)$ where w_2 denotes the second Stiefel-Whitney class, and $c_1(L)$ is the first Chern class of L. Then the vector bundle U has a Spin^c-structure. Let h^V, h^L be metrics on V, L. Let $S(U, L)$ be the fundamental complex spinor bundle for (U, L) [**LaM**, Appendix D.9].

Assume that the S^1-action on M lifts to V, L and W, and assume the metrics h^{TX}, h^V, h^L, h^W are S^1-invariant. Also assume that the S^1-actions on TX, V, L lifts to $S(U, L)$.

Let ∇^{TX} be the Levi-Civita connection on (TX, h^{TX}) along the fibre X. Let ∇^V, ∇^L and ∇^W be S^1 invariant and metric-compatible connections on (V, h^V), (L, h^L) and (W, h^W) respectively. Let $\nabla^{S(U,L)}$ be the Hermitian connection on $S(U, L)$ induced by $\nabla^{TX} \oplus \nabla^V$ and ∇^L as in Section 1.1. Let $\nabla^{S(U,L) \otimes W}$ be the tensor product connection on $S(U, L) \otimes W$ induced by $\nabla^{S(U,L)}$ and ∇^W.

Let $\{e_i\}_{i=1}^{2l}$, $\{f_j\}_{j=1}^{2p}$ be the corresponding oriented orthonormal basis of (TX, h^{TX}) and (V, h^V). Let D^X be the family Spin^c-Dirac operator on the fiber X,

$$(1.44) \qquad D^X = \sum_{i=1}^{2l} c(e_i) \nabla^{S(U,L) \otimes W}_{e_i}.$$

There are two canonical ways to consider $S(U, L)$ as a $\mathbf{Z}_2$-graded vector bundle. Let

$$(1.45) \qquad \begin{aligned} \tau_s &= i^l c(e_1) \cdots c(e_{2l}), \\ \tau_e &= i^{l+p} c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}) \end{aligned}$$

be two involutions of $S(U, L)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(U, L) = S^+(U, L) \oplus S^-(U, L)$ corresponding to τ_s (resp. τ_e) such that $\tau_s|_{S^\pm(U,L)} = \pm 1$ (resp. $\tau_e|_{S^\pm(U,L)} = \pm 1$). For $\tau = \tau_s$ or τ_e, we can define the index bundle $\mathrm{Ind}_\tau(D^X) \in K_{S^1}(B)$ as in Section 1.1.

We have the following S^1-equivariant decomposition of V restricted to F,

$$(1.46) \qquad V_{|F} = \oplus_{v \neq 0} V_v \oplus V_0^{\mathbf{R}},$$

where V_v is a complex vector bundle such that g acts on it by g^v, and $V_0^{\mathbf{R}}$ is the real subbundle of V such that S^1 acts as identity. For $v \neq 0$, let $V_{v,\mathbf{R}}$ denote the underlying real vector bundle of V_v. Denote by $2p' = \dim V_0^{\mathbf{R}}$ and $2l' = \dim Y$.

Let us write

$$(1.47) \qquad L_F = L \otimes \left(\bigotimes_{v \neq 0} \det N_v \bigotimes_{v \neq 0} \det V_v \right)^{-1}.$$

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Then $TY \oplus V_0^{\mathbf{R}}$ has a Spin^c structure as $w_2(TY \oplus V_0^{\mathbf{R}}) = c_1(L_F)$ mod (2). Let $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ be the fundamental spinor bundle for $(TY \oplus V_0^{\mathbf{R}}, L_F)$.

Let D^Y, D^{Y_α} be the families of Spin^c Dirac operators acting on $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ over F, F_α. If R is an Hermitian complex vector bundle equipped with an Hermitian connection over F, let $D^Y \otimes R$, $D^{Y_\alpha} \otimes R$ denote the twisted Spin^c Dirac operators on $S(TY \oplus V_0^{\mathbf{R}}, L_F) \otimes R$.

Recall that $N_{v,\mathbf{R}}$ and $V_{v,\mathbf{R}}$ are canonically oriented by their complex structures. The decompositions (1.8), (1.46) induce the orientations of TY and $V_0^{\mathbf{R}}$. Let $\{e_i\}_{i=1}^{2l'}$, $\{f_j\}_{j=1}^{2p'}$ be the corresponding oriented orthonormal basis of (TY, h^{TY}) and $(V_0^{\mathbf{R}}, h^{V_0^{\mathbf{R}}})$. There are two canonical ways to consider $S(TY \oplus V_0^{\mathbf{R}}, L_F)$ as a $\mathbf{Z}_2$-graded vector bundle: let

$$(1.48) \qquad \begin{aligned} \tau_s &= i^{l'} c(e_1) \cdots c(e_{2l'}), \\ \tau_e &= i^{l'+p'} c(e_1) \cdots c(e_{2l'}) c(f_1) \cdots c(f_{2p'}) \end{aligned}$$

be two involutions of $S(TY \oplus V_0^{\mathbf{R}}, L_F)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(TY \oplus V_0^{\mathbf{R}}, L_F) = S^+(TY \oplus V_0^{\mathbf{R}}, L_F) \oplus S^-(TY \oplus V_0^{\mathbf{R}}, L_F)$ corresponding to τ_s (resp. τ_e) such that $\tau_s|_{S^\pm(TY \oplus V_0^{\mathbf{R}}, L_F)} = \pm 1$ (resp. $\tau_e|_{S^\pm(TY \oplus V_0^{\mathbf{R}}, L_F)} = \pm 1$).

By restricting to F, one has the following isomorphism of $\mathbf{Z}_2$-graded Clifford modules over F,

$$(1.49) \qquad S(U, L) \simeq S\left(TY \oplus V_0^{\mathbf{R}}, L_F\right) \widehat{\bigotimes_{v \neq 0}} \Lambda N_v \widehat{\bigotimes_{v \neq 0}} \Lambda V_v.$$

We denote by Ind_{τ_s}, Ind_{τ_e} the index bundles corresponding to the involutions τ_s, τ_e.

Let S^1 act on L by send $g \in S^1$ to g^{l_c} ($l_c \in \mathbf{Z}$) on F. Then l_c is locally constant on F. We define the following elements in $K(F)[[q^{1/2}]]$,

$$(1.50) \qquad \begin{aligned} R_\pm(q) &= q^{\frac{1}{2}\Sigma_v |v| \dim N_v - \frac{1}{2}\Sigma_v v \dim V_v + \frac{1}{2} l_c} \otimes_{0<v} \left(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\right) \\ &\quad \otimes_{v<0} \mathrm{Sym}_{q^{-v}}\left(\overline{N}_v\right) \otimes_{v \neq 0} \Lambda_{\pm q^v}(V_v) \otimes_v q^v W_v = \sum_n R_{\pm,n} q^n, \\ R'_\pm(q) &= q^{-\frac{1}{2}\Sigma_v |v| \dim N_v - \frac{1}{2}\Sigma_v v \dim V_v + \frac{1}{2} l_c} \otimes_{0<v} \mathrm{Sym}_{q^{-v}}\left(\overline{N}_v\right) \\ &\quad \otimes_{v<0}\left(\mathrm{Sym}_{q^v}(N_v) \otimes \det N_v\right) \otimes_{v \neq 0} \Lambda_{\pm q^v}(V_v) \otimes_v q^v W_v = \sum_n R'_{\pm,n} q^n. \end{aligned}$$

The following result generalizes [**T**, Theorem 2.6] to the family case.

Theorem 1.2 *For $n \in \mathbf{Z}$, we have the following identity in $K(B)$,*

$$(1.51) \qquad \begin{aligned} \mathrm{Ind}_{\tau_s}\left(D^X, n\right) &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}_{\tau_s}\left(D^{Y_\alpha} \otimes R_{+,n}\right) \\ &= \sum_\alpha (-1)^{\Sigma_{v<0} \dim N_v} \mathrm{Ind}_{\tau_s}\left(D^{Y_\alpha} \otimes R'_{+,n}\right), \\ \mathrm{Ind}_{\tau_e}\left(D^X, n\right) &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}_{\tau_e}\left(D^{Y_\alpha} \otimes R_{-,n}\right) \\ &= \sum_\alpha (-1)^{\Sigma_{v<0} \dim N_v} \mathrm{Ind}_{\tau_e}\left(D^{Y_\alpha} \otimes R'_{-,n}\right). \end{aligned}$$

Proof: The proof is a straightforward generalization of the proof of Theorem 1.1. The details are left to the interested reader. ∎

2 Family rigidity and vanishing theorems

The purpose of this section is to establish the main results of this paper: the rigidity and vanishing theorems on the equivariant K-theory level. The results in this section refine some of the results in [**LiuMa1, 2**] to the K-theory level.

As in the previous sections, we let $\pi : M \to B$ be a fibration of compact manifolds with fiber X. We assume that S^1 acts fiberwise on M with fixed point set F, and TX has an S^1-equivariant spin structure. Then $\pi : F \to B$ is a fibration with fiber Y.

Following Witten, we will introduce some elements $R(q) = \Sigma_{n \in \mathbf{N}} q^n R_n \in K_{S^1}(M)[[q]]$. To prove the rigidity theorem for these elements, Taubes and Witten suggested to use some shift operators to get a relation like $\mathrm{Ind}(D^X \otimes R_m, h) = \mathrm{Ind}(D^X \otimes R_{m+h}, h)$ for $h, m \in \mathbf{Z}$. As $R_m = 0$ for $m < 0$, this implies the rigidity theorem. See the paper of Taubes [**T**] for a rigorous treatment.

To get a similar equality in the family case, we first apply our K-theory version of the equivariant family index theorem [**LiuMa1**, Theorem 1.1], Theorem 1.1, to reduce the problem to the fixed point set F. Then we introduce an auxiliary element in $K_{S^1}(F)[[q]]$. We study the corresponding index bundles of the twisted Spin^c Dirac operators on Y, which, after doing some shift operations, will be related to a term like $\mathrm{Ind}(D^X \otimes R_{m+h}, h)$. On the other hand, if we apply our K-theory version of the equivariant family index theorem iterately, we may also relate the considered index bundle to a term like $\mathrm{Ind}(D^X \otimes R_m, h)$. This then completes the proof. To apply the equivariant family index theorem, we are inspired by the constructions of Taubes [**T**, §6]. Namely, we will construct some operators on the fixed point set $M(n)$ of the induced $\mathbf{Z}_n$-action on M, and apply the equivariant family index theorem to them.

As was pointed out in the introduction, our main observation is that we can directly construct and apply the shift operators on the fixed point set. In this way, we avoid the use of the Dirac operators on the normal bundle in the loop space and the associated analysis of Fredholm properties of these operators in [**T**].

This section is organized as follows: In Section 2.1, we state our main results, the rigidity and vanishing theorems on the equivariant K-theory level. In Section 2.2, we state two intermediate results which will be used to prove our main results stated in Section 2.1. In Section 2.3, we prove the family rigidity and vanishing theorem. In Section 2.4, we prove a vanishing theorem on the equivariant K-theory level of the index bundle of the Dirac operator on loop spaces, which may be viewed as a loop space analogue of Corollary 1.1, and which also extends the corresponding loop space analogue of the Atiyah-Hirzebruch theorem in [**Liu2**] to the family case.

Throughout this section, we use the notations of Section 1.1.

2.1 Family rigidity and vanishing theorems

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X and $\dim X = 2l$. We assume that S^1 acts fiberwise on M, and TX has an S^1-invariant spin structure. As in [**AH**], by lifting to the double cover of S^1, we can assume that the second condition is always satisfied. Let V be a real vector bundle over M with structure group $\mathrm{Spin}(2k)$. We assume that V has an S^1-invariant spin structure.

The purpose of this section is to prove that the elliptic operators introduced by Witten [**W**] have some interesting rigidity and vanishing properties on the equivariant K-theory level. Let us recall some definitoins first.

For a complex (resp. real) vector bundle E over M, let

(2.1)
$$\mathrm{Sym}_t(E) = 1 + tE + t^2\mathrm{Sym}^2 E + \cdots,$$
$$\Lambda_t(E) = 1 + tE + t^2\Lambda^2 E + \cdots$$

be the symmetric and exterior power operations of E (resp. $E \otimes_{\mathbf{R}} \mathbf{C}$) in $K(M)[[t]]$ respectively. Set

(2.2)
$$\Theta'_q(TX) = \otimes_{n=1}^{\infty}\Lambda_{q^n}(TX) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX),$$
$$\Theta_q(TX) = \otimes_{n=1}^{\infty}\Lambda_{-q^{n-1/2}}(TX) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX),$$
$$\Theta_{-q}(TX) = \otimes_{n=1}^{\infty}\Lambda_{q^{n-1/2}}(TX) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX).$$

We also define the following elements in $K(M)[[q^{1/2}]]$:

(2.3)
$$\Theta'_q(TX|V) = \otimes_{n=1}^{\infty}\Lambda_{q^n}(V) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX),$$
$$\Theta_q(TX|V) = \otimes_{n=1}^{\infty}\Lambda_{-q^{n-1/2}}(V) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX),$$
$$\Theta_{-q}(TX|V) = \otimes_{n=1}^{\infty}\Lambda_{q^{n-1/2}}(V) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX),$$
$$\Theta^*_q(TX|V) = \otimes_{n=1}^{\infty}\Lambda_{-q^n}(V) \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX).$$

Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of V.

Recall that the equivariant cohomology group $H^*_{S^1}(M, \mathbf{Z})$ of M is defined by

(2.4)
$$H^*_{S^1}(M, \mathbf{Z}) = H^*(M \times_{S^1} ES^1, \mathbf{Z}),$$

where ES^1 is the usual universal S^1-principal bundle over the classifying space BS^1 of S^1. So $H^*_{S^1}(M, \mathbf{Z})$ is a module over $H^*(BS^1, \mathbf{Z})$ induced by the projection $\overline{\pi} : M \times_{S^1} ES^1 \to BS^1$. Let $p_1(V)_{S^1}$, $p_1(TX)_{S^1} \in H^*_{S^1}(M, \mathbf{Z})$ be the equivariant first Pontrjagin classes of V and TX respectively. As $V \times_{S^1} ES^1$ and $TX \times_{S^1} ES^1$ are spin over $M \times_{S^1} ES^1$, one knows that $\frac{1}{2}p_1(V)_{S^1}$ and $\frac{1}{2}p_1(TX)_{S^1}$ are well-defined in $H^*_{S^1}(M, \mathbf{Z})$ (cf. [**T**, pp. 456-457]). Also recall that

(2.5)
$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[[u]]$$

with u a generator of degree 2.

In the following, we denote by $D^X \otimes W$ the family of Dirac operators acting fiberwise on $S(TX) \otimes W$ as defined in Section 1.1. We also write $d^X_s = D^X \otimes S(TX)$.

We can now state the main results of this paper as follows. The first one is a family generalization of the Witten rigidity theorems as being proved in [**T**], [**BT**] and [**Liu1**]. It also refines [**LiuMa1**, Theorem 2.1].

Theorem 2.1 *(a) The family operators $d^X_s \otimes \Theta'_q(TX)$, $D^X \otimes \Theta_q(TX)$ and $D^X \otimes \Theta_{-q}(TX)$ are rigid on the equivariant K-theory level.*

(b) If $\frac{1}{2}p_1(V)_{S^1} = \frac{1}{2}p_1(TX)_{S^1}$, then $D^X \otimes (S^+(V) + S^-(V)) \otimes \Theta'_q(TX|V)$, $D^X \otimes (S^+(V) - S^-(V)) \otimes \Theta^*_q(TX|V)$, $D^X \otimes \Theta_q(TX|V)$ and $D^X \otimes \Theta_{-q}(TX|V)$ are rigid on the equivariant K-theory level.

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The second one generalizes the vanishing results of Taubes [**T**, Proposition 10.1] and Liu [**Liu2**, Corollary 3.3] to the family case. It also refines a result in [**LiuMa1**, Theorem 3.2].

Theorem 2.2 *If $\frac{1}{2}p_1(V)_{S^1} - \frac{1}{2}p_1(TX)_{S^1} = e \cdot \overline{\pi}^* u^2$ with $e \in \mathbf{Z}$ verifying $e < 0$, then the index bundles of $D^X \otimes (S^+(V) + S^-(V)) \otimes \Theta'_q(TX|V)$, $D^X \otimes (S^+(V) - S^-(V)) \otimes \Theta^*_q(TX|V)$, $D^X \otimes \Theta_q(TX|V)$ and $D^X \otimes \Theta_{-q}(TX|V)$ are zero in $K_{S^1}(B)$. In particular, they are identically zero in $K(B)$.*

A quite interesting consequence of the above results is the following family $\widehat{A}$-vanishing theorem for loop spaces. It extends the corresponding loop space analogue of the Atiyah-Hirzebruch theorem in [**Liu2**, Theorem 6] to the family case.

Theorem 2.3 *Assume M is connected and the S^1-action is nontrivial. If $\frac{1}{2}p_1(TX)_{S^1} = -e \cdot \overline{\pi}^* u^2$ for some integer e, then the index bundle of $D^X \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX)$ as an element in $K_{S^1}(B)$, in particular, as an element in $K(B)$, is identically zero.*

As was pointed out by Dessai [**De**], when the S^1-action is induced from a fiberwise S^3 action on M which preserves the Spin structure of TX, the condition $\frac{1}{2}p_1(TX)_{S^1} = -e \cdot \overline{\pi}^* u^2$ in $H^*_{S^1}(M, \mathbf{Z})$ is equivalent to $\frac{1}{2}p_1(TX) = 0$ in $H^*(M, \mathbf{Z})$. So one gets the following family vanishing theorem.

Corollary 2.1 *Assume M is connected and admits a nontrivial S^1 action induced by a fiberwise S^3-action which preserves the spin structure of TX. If $\frac{1}{2}p_1(TX) = 0$, then the index bundle as an element in $K_{S^1}(B)$ (in particular, as an element in $K(B)$), of $D^X \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX)$, is identically zero.*

Actually, our proof of these Theorems works under the following slightly weaker hypothesis. Let us first explain some notations.

For each $n > 1$, consider $\mathbf{Z}_n \subset S^1$, the cyclic subgroup of order n. We have the $\mathbf{Z}_n$ equivariant cohomology of M defined by $H^*_{\mathbf{Z}_n}(M, \mathbf{Z}) = H^*(M \times_{\mathbf{Z}_n} ES^1, \mathbf{Z})$, and there is a natural "forgetful" map $\alpha(S^1, \mathbf{Z}_n) : M \times_{\mathbf{Z}_n} ES^1 \to M \times_{S^1} ES^1$ which induces a pullback $\alpha(S^1, \mathbf{Z}_n)^* : H^*_{S^1}(M, \mathbf{Z}) \to H^*_{\mathbf{Z}_n}(M, \mathbf{Z})$. The arrow which forgets the S^1 action altogether we denote by $\alpha(S^1, 1)$. Thus $\alpha(S^1, 1)^* : H^*_{S^1}(M, \mathbf{Z}) \to H^*(M, \mathbf{Z})$ is induced by the inclusion of M into $M \times_{S^1} ES^1$ as a fiber over BS^1.

Finally, note that, if $\mathbf{Z}_n$ acts trivially on a space Y, then there is another map $r^* : H^*(Y, \mathbf{Z}) \to H^*_{\mathbf{Z}_n}(Y, \mathbf{Z})$ induced by the projection $Y \times_{\mathbf{Z}_n} ES^1 = Y \times B\mathbf{Z}_n \xrightarrow{r} Y$.

We let $\mathbf{Z}_\infty = S^1$. For each $1 < n \leq \infty$, let $i : M(n) \to M$ be the inclusion of the fixed point set of $\mathbf{Z}_n \subset S^1$ in M and so i induces $i_{S^1} : M(n) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$.

In the rest of this paper, we suppose that there exists some integer $e \in \mathbf{Z}$ such that for each $1 < n \leq \infty$,

$$
\begin{aligned}
(2.6) \qquad & \alpha\left(S^1, \mathbf{Z}_n\right)^* \circ i_{S^1}^* \left(\tfrac{1}{2}p_1(V - TX)_{S^1} - e \cdot \overline{\pi}^* u^2\right) \\
& = r^* \circ \alpha\left(S^1, 1\right)^* \circ i_{S^1}^* \left(\tfrac{1}{2}p_1(V - TX)_{S^1}\right).
\end{aligned}
$$

Remark 2.1. The relation (2.6) clearly follows from the hypothesis of Theorems 2.1 and 2.2 by pulling back and forgetting. Thus it is a weaker condition.

Remark 2.2. If $e = 0$, and B is a point, (2.6) is exactly [**BT**, (11.11)].

We can now state a slightly more general version of Theorems 2.1 and 2.2.

Theorem 2.4 *Under the hypothesis (2.6), we have*
i) If $e = 0$, then the index bundles of the elliptic operators in Theorem 2.1(b) are rigid on the equivariant K-theory level.
ii) If $e < 0$, then the index bundles of the elliptic operators in Theorem 2.1(b) are zero as elements in $K_{S^1}(B)$. In particular, these index bundles are zero in $K(B)$.

The rest of this section is devoted to a proof of Theorem 2.4.

2.2 Two intermediate results

Let $F = \{F_\alpha\}$ be the fixed point set of the circle action. Then $\pi : F \to B$ is a fibration with compact fibre denoted by $Y = \{Y_\alpha\}$. Recall that

$$TX_{|F} = TY \oplus_{v \neq 0} N_{v,\mathbf{R}},$$

where $N_{v,\mathbf{R}}$ denotes the underlying real bundle of the complex vector bundle N_v on which $g \in S^1$ acts by multiplying by g^v. Since we can choose either N_v or $\overline{N}_v$ as the complex vector bundle for $N_{v,\mathbf{R}}$, in what follows we may and we will assume that

$$(2.7) \qquad \begin{aligned} TX_{|F} &= TY \oplus \bigoplus_{0 < v} N_v, \\ TX \otimes_{\mathbf{R}} \mathbf{C} &= TY \otimes_{\mathbf{R}} \mathbf{C} \bigoplus_{0 < v} (N_v \oplus \overline{N}_v), \end{aligned}$$

where N_v is the complex vector bundle on which $g \in S^1$ acts by multiplying by g^v. Here some N_v may be zero. Similarly, let

$$(2.8) \qquad V_{|F} = V_0^{\mathbf{R}} \oplus \bigoplus_{0 < v} V_v,$$

where V_v is a complex vector bundle on which $g \in S^1$ acts by multiplying by g^v, and $V_0^{\mathbf{R}}$ is a real vector bundle on which S^1 acts as identity.

On F, let

$$(2.9) \qquad \begin{aligned} e(N) &= \sum_{0<v} v^2 \dim N_v, & d'(N) &= \sum_{0<v} v \dim N_v, \\ e(V) &= \sum_{0<v} v^2 \dim V_v, & d'(V) &= \sum_{0<v} v \dim V_v. \end{aligned}$$

Then $e(N)$, $e(V)$, $d'(N)$ and $d'(V)$ are locally constant functions on F.

Let us introduce some line bundles:

$$(2.10) \qquad \begin{aligned} L(N) &= \otimes_{0<v} (\det N_v)^v, & L(V) &= \otimes_{0<v} (\det V_v)^v, \\ L &= L(N)^{-1} \otimes L(V). \end{aligned}$$

We denote the Chern roots of N_v by $\{x_v^j\}$ (resp. V_v by u_v^j), and the Chern roots of $TY \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm y_j\}$ (resp. $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ by $\{\pm u_0^j\}$). Then if we take $\mathbf{Z}_\infty = S^1$ in (2.6),

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we get

$$p_1(V)_{S^1} = \Sigma_{v,j} \left(u_v^j + vu\right)^2,$$
$$p_1(TX)_{S^1} = \Sigma_j(y_j)^2 + \Sigma_{v,j}\left(x_v^j + vu\right)^2,$$
(2.11)
$$\frac{1}{2}\left(\Sigma_{v,j}\left(u_v^j + vu\right)^2 - \Sigma_j(y_j)^2 - \Sigma_{v,j}\left(x_v^j + vu\right)^2\right) - eu^2$$
$$= \frac{1}{2}\left(\Sigma_{v,j}\left(u_v^j\right)^2 - \Sigma_j(y_j)^2 - \Sigma_{v,j}\left(x_v^j\right)^2\right).$$

By (2.5), (2.11), we get

(2.12)
$$c_1(L) = \Sigma_{v,j} vu_v^j - \Sigma_{v,j} vx_v^j = 0,$$
$$e(V) - e(N) = \sum_{0<v} v^2 \dim V_v - \sum_{0<v} v^2 \dim N_v = 2e.$$

This means L is a trivial complex line bundle over each component F_α of F, and S^1 acts on L by multiplying by g^{2e}. So we can extend L to a trivial complex line bundle over M, and we extend the S^1 action on it by multiplying the canonical section 1 of L to $g^{2e} \cdot 1$.

The line bundles in (2.10) will play important roles in the next two sections which contain the proofs of Theorems 2.5, 2.6 to be stated below.

We now define the following elements in $K_{S^1}(M)[[q^{1/2}]]$:

(2.13)
$$R_1(V) = (S^+(V) + S^-(V)) \otimes_{n=1}^\infty \Lambda_{q^n}(V),$$
$$R_2(V) = (S^+(V) - S^-(V)) \otimes_{n=1}^\infty \Lambda_{-q^n}(V),$$
$$R_3(V) = \otimes_{n=1}^\infty \Lambda_{-q^{n-1/2}}(V),$$
$$R_4(V) = \otimes_{n=1}^\infty \Lambda_{q^{n-1/2}}(V).$$

In what follows, if $R(q) = \sum_{m\in\frac{1}{2}\mathbf{Z}} R_m q^m \in K_{S^1}(M)[[q^{1/2}]]$, we will also denote $\mathrm{Ind}(D^X \otimes R_m, h)$ by $\mathrm{Ind}(D^X \otimes R(q), m, h)$.

We first state a result which expresses the global equivariant family index via the family indices on the fixed point set.

Proposition 2.1 *For $m \in \frac{1}{2}\mathbf{Z}$, $h \in \mathbf{Z}$, $1 \le i \le 4$, we have the following identity in $K(B)$,*

(2.14)
$$\mathrm{Ind}\left(D^X \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes R_i(V), m, h\right)$$
$$= \sum_\alpha (-1)^{\Sigma_{0<v}\dim N_v}\mathrm{Ind}(D^{Y_\alpha} \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes R_i(V)$$
$$\otimes \mathrm{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v, m, h)$$

Proof. Proposition 2.1 follows directly from Theorem 1.1. ■

For $p \in \mathbf{N}$, we define the following elements in $K_{S^1}(F)[[q]]$:

$$\mathcal{F}_p(X) = \bigotimes_{0<v}\left(\otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(N_v) \otimes_{n>pv} \mathrm{Sym}_{q^n}(\overline{N}_v)\right) \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TY),$$
(2.15)
$$\mathcal{F}'_p(X) = \bigotimes_{\substack{0<v\\0\le n\le pv}}\left(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v\right),$$

$$\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_p(X).$$

Then, by (2.7), over F, we have

$$(2.16) \qquad \mathcal{F}^0(X) = \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes \mathrm{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v.$$

We now state two intermediate results on the relations between the family indices on the fixed point set. They will be used in the next subsection to prove Theorem 2.4.

Theorem 2.5 *For $1 \le i \le 4$, h, $p \in \mathbf{Z}$ and $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, we have*

$$(2.17) \qquad \begin{aligned} &\sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_i(V), m, h) \\ &= \sum_\alpha (-1)^{\Sigma_{0<v} \dim N_v} \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_i(V), \\ &\qquad\qquad m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h). \end{aligned}$$

Theorem 2.6 *For each α, $1 \le i \le 4$, h, $p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, we have the following identity in $K(B)$,*

$$(2.18) \qquad \begin{aligned} &\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_i(V), m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h) \\ &= \mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes R_i(V), m + ph + p^2 e, h + 2pe). \end{aligned}$$

Theorem 2.5 is a direct consequence of Theorem 2.7 to be stated below, which will be proved in Section 4, while Theorem 2.6 will be proved in Section 3.

To state Theorem 2.7, set $J = \{v \in \mathbf{N}|$ there exists α such that $N_v \ne 0$ on $F_\alpha\}$ and

$$(2.19) \qquad \Phi = \{\beta \in]0,1] | \text{there exists } v \in J \text{ such that } \beta v \in \mathbf{Z}\}.$$

We order the elements in Φ so that $\Phi = \{\beta_i | 1 \le i \le J_0, J_0 \in \mathbf{N} \text{ and } \beta_i < \beta_{i+1}\}$. Then for any integer $1 \le i \le J_0$, there exist p_i, $n_i \in \mathbf{N}$, $0 < p_i \le n_i$ with $(p_i, n_i) = 1$ such that

$$(2.20) \qquad \beta_i = p_i/n_i.$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $1 \le j \le J_0$, $p \in \mathbf{N}^*$, set

$$(2.21) \qquad \begin{aligned} &I_0^p = \phi, \text{ the empty set}, \\ &I_j^p = \{(v,n)|v \in J, (p-1)v < n \le pv, \tfrac{n}{v} = p - 1 + \tfrac{p_j}{n_j}\}, \\ &\overline{I}_j^p = \{(v,n)|v \in J, (p-1)v < n \le pv, \tfrac{n}{v} > p - 1 + \tfrac{p_j}{n_j}\}. \end{aligned}$$

For $0 \le j \le J_0$, we write

$$(2.22)$$
$$\mathcal{F}_{p,j}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \bigotimes_{(v,n)\in\cup_{i=1}^j I_i^p} \left(\mathrm{Sym}_{q^{-n}}(N_v) \otimes \det N_v \right) \bigotimes_{(v,n)\in\overline{I}_j^p} \mathrm{Sym}_{q^n}(\overline{N}_v).$$

Then

$$(2.23) \qquad \begin{aligned} \mathcal{F}_{p,0}(X) &= \mathcal{F}^{-p+1}(X), \\ \mathcal{F}_{p,J_0}(X) &= \mathcal{F}^{-p}(X). \end{aligned}$$

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For $s \in \mathbf{R}$, let $[s]$ denote the greatest integer which is less than or equal to the given number s. Set

$$(2.24) \quad \begin{aligned} e(p, \beta_j, N) &= \tfrac{1}{2} \sum_{0<v} (\dim N_v) \Big((p-1)v + [\tfrac{p_j v}{n_j}] \Big) \Big((p-1)v + [\tfrac{p_j v}{n_j}] + 1 \Big), \\ d'(\beta_j, N) &= \sum_{0<v} (\dim N_v)[\tfrac{p_j v}{n_j}]. \end{aligned}$$

Then $e(p, \beta_j, N), d'(\beta_j, N)$ are locally constant functions on F. And

$$(2.25) \quad \begin{aligned} e(p, \beta_0, N) &= \tfrac{1}{2}(p-1)^2 e(N) + \tfrac{1}{2}(p-1)d'(N), \\ e(p, \beta_{J_0}, N) &= \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N). \end{aligned}$$

Theorem 2.7 *For $1 \leq i \leq 4$, $1 \leq j \leq J_0$, $p \in \mathbf{N}^*$, $h \in \mathbf{Z}$, $m \in \tfrac{1}{2}\mathbf{Z}$, we have*

$$(2.26) \quad \begin{aligned} \sum_\alpha (-1)^{d'(\beta_{j-1},N)+\Sigma_{0<v}\dim N_v} &\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}_{p,j-1}(X) \otimes R_i(V), \\ & m + e(p, \beta_{j-1}, N), h) \\ = \sum_\alpha (-1)^{d'(\beta_j,N)+\Sigma_{0<v}\dim N_v} &\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}_{p,j}(X) \otimes R_i(V), \\ & m + e(p, \beta_j, N), h). \end{aligned}$$

Proof: The proof is given in Section 4. ∎

Proof of Theorem 2.5 : From (2.23), (2.25), and Theorem 2.7, for $1 \leq i \leq 4$, $h \in \mathbf{Z}$, $p \in \mathbf{N}^*$ and $m \in \tfrac{1}{2}\mathbf{Z}$, we get the following identity in $K(B)$:

$$(2.27) \quad \begin{aligned} \sum_\alpha (-1)^{\Sigma_{0<v}(v+1)\dim N_v} &\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_i(V), \\ & m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h) \\ = \sum_\alpha (-1)^{\Sigma_{0<v}\dim N_v} &\mathrm{Ind}(D^{Y_\alpha} \otimes \mathcal{F}^{-p+1}(X) \otimes R_i(V), \\ & m + \tfrac{1}{2}(p-1)^2 e(N) + \tfrac{1}{2}(p-1)d'(N), h). \end{aligned}$$

As $\sum_{0<v} v \dim N_v = 0 \bmod (2)$, we get Theorem 2.5. ∎

2.3 Proof of Theorem 2.4

From Proposition 2.1, Theorems 2.5 and 2.6, for $1 \leq i \leq 4$, $h, p \in \mathbf{Z}$, $p > 0$ and $m \in \tfrac{1}{2}\mathbf{Z}$, we get the following identity in $K(B)$,

$$(2.28) \quad \begin{aligned} \mathrm{Ind}(D^X \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes R_i(V), m, h) \\ = \mathrm{Ind}(D^X \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes R_i(V), m', h + 2pe). \end{aligned}$$

with

$$(2.29) \quad m' = m + ph + p^2 e.$$

Note that, from (2.1) and (2.13), if $m < 0$, for $h \in \mathbf{Z}$, we have

$$(2.30) \quad \mathrm{Ind}(D^X \otimes_{n=1}^\infty \mathrm{Sym}_{q^n}(TX) \otimes R_i(V), m, h) = 0 \quad \text{in} \quad K(B).$$

i) Assume that $e = 0$. Let $h \in \mathbf{Z}$, $m_0 \in \tfrac{1}{2}\mathbf{Z}$ and $h \neq 0$ be fixed. If $h > 0$, we take $m' = m_0$, then for p big enough, we get $m < 0$ in (2.29). If $h < 0$, we take $m = m_0$,

then for p big enough, we get $m' < 0$ in (2.29). From (2.30), we know that for $h \neq 0$, $m_0 \in \frac{1}{2}\mathbf{Z}$ and $1 \leq i \leq 4$, we get

$$(2.31) \qquad \mathrm{Ind}(D^X \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TX) \otimes R_i(V), m_0, h) = 0 \quad \mathrm{in} \quad K(B).$$

ii) Assume that $e < 0$. For $h \in \mathbf{Z}$, $m_0 \in \frac{1}{2}\mathbf{Z}$, $1 \leq i \leq 4$, we take $m = m_0$, then for p big enough, we get $m' < 0$ in (2.29). From (2.30), we again get (2.31).

The proof of Theorem 2.4 is complete. ∎

Remark 2.3. It might be suitable to add a remark on the comparison of the various proofs of the Witten rigidity theorem given in [**T**], [**BT**], [**Liu1**] and the present paper. On one hand, the proofs in [**BT**] and [**Liu1**] relies on the Atiyah-Bott-Segal-Singer fixed point formula (cf. [**ASe**], [**ASi**]) and the elliptic function theory, so they don't work on the K-theory level. This is reflected in [**LiuMa1**] where a proof of a family Witten rigidity theorem on the equivariant Chern character level is given by extending the method in [**Liu1**]. On the other hand, consider the proofs given in [**T**], [**BT**] and the present paper (for the last see Section 4 for more details). All these proofs rely on Taubes's idea that in certain steps one needs to consider Dirac operators on the fixed point set of the induced $\mathbf{Z}_n$-actions. This requires that the topological conditions imposed in these proofs are for the half of the equivariant first Pontryagin classes. While the proof in [**Liu1**] works directly on the fixed point set of the S^1-action, and thus works under weaker conditions on the equivariant first Pontryagin classes without the factor $\frac{1}{2}$. This leads to the natural question that whether there would exist a K-theory version of the proof in [**Liu1**]. Very likely one needs to use Hecke operator in the theory of modular forms to understand the shift operators and the modular transformations.

2.4 Proof of Theorem 2.3

In fact, by setting $V = 0$ in (2.12), we know that

$$(2.32) \qquad \Sigma_{0 < v} v^2 \dim N_v = -2e.$$

Thus the case $e > 0$ can never happen. If $e = 0$, then all the numbers $\dim N_v$ are zero, so that the S^1-action cannot have fixed points. From Theorem 1.1, we know that the index bundle is zero in $K_{S^1}(B)$. For $e < 0$, one may take $V = 0$ in Theorem 2.4 to derive Theorem 2.3.

The proof of Theorem 2.3 is complete. ∎

3   Proof of Theorem 2.6

In this section, we will prove Theorem 2.6 by introducing some shift operators as in [**T**, §7].

    This section is organized as follows: In Section 3.1, we introduce some notations. In Section 3.2, we prove Theorem 2.6 by introducing some shift operators as in [**T**, §7].

    Throughout this section, we use the notations of Section 2.

3.1   Reformulation of Theorem 2.6

To simplify the notations, we introduce some new notations in this subsection.

    For $n_0 \in \mathbf{N}^*$, we define a number operator P on $K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$ in the following way: if $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbf{Z}} q^n R_n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, then P acts on $R(q)$ by multiplication by n on R_n. From now on, we simply write $\mathrm{Sym}_{q^n}(TX)$, $\Lambda_{q^n}(V)$ as $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ respectively. In this way, P acts on TX_n and V_n by multiplication by n, and the actions P on $\mathrm{Sym}(TX_n)$, $\Lambda(V_n)$ are naturally induced from its actions on TX_n and V_n, etc. So the eigenspace of $P = n$ is given by the coefficient of q^n of the corresponding element $R(q)$. For $R(q) = \oplus_{n \in \frac{1}{n_0}\mathbf{Z}} q^n R_n \in K_{S^1}(M)[[q^{\frac{1}{n_0}}]]$, we will also write

$$(3.1) \qquad \mathrm{Ind}(D^X \otimes R(q), m, h) = \mathrm{Ind}(D^X \otimes R_m, h).$$

    Let H be the canonical basis of $\mathrm{Lie}(S^1) = \mathbf{R}$, i.e., $\exp(tH) = \exp(2\pi i t)$ for $t \in \mathbf{R}$. If E is an S^1-equivariant vector bundle over M, let L_H denote the correponding infinitesimal action of H on $\Gamma(M, E)$.

    On the fixed point set F, let J_H be the representation of $\mathrm{Lie}(S^1)$ on $E|_F$. Then on $\Gamma(F, E|_F)$, L_H is exactly the operator J_H on $\Gamma(F, E|_F)$, and the weight of the S^1 action on $\Gamma(F, E|_F)$ is given by the action

$$(3.2) \qquad \mathbf{J}_H = \frac{-1}{2\pi}\sqrt{-1}J_H.$$

    Recall that the $\mathbf{Z}_2$ grading on $S(TX) \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n)$ (resp. $S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X)$) is induced by the $\mathbf{Z}_2$-grading on $S(TX)$ (resp. $S(TY, \otimes_{0<v}(\det N_v)^{-1})$). Write

$$(3.3) \qquad \begin{aligned} F_V^1 &= S(V) \bigotimes_{n=1}^{\infty} \Lambda(V_n), \\ F_V^2 &= \otimes_{n \in \mathbf{N}+\frac{1}{2}} \Lambda(V_n). \end{aligned}$$

There are two natural $\mathbf{Z}_2$ gradings on F_V^1, F_V^2. The first grading is induced by the $\mathbf{Z}_2$-grading of $S(V)$ and the nature $\mathbf{Z}_2$ grading (induced by forms of homogeneous degree) of $\bigotimes_{n=1}^{\infty} \Lambda(V_n)$ and $\otimes_{n \in \mathbf{N}+\frac{1}{2}} \Lambda(V_n)$. We define $\tau_{e|F_V^{i\pm}} = \pm 1$ to be the involution defined by this $\mathbf{Z}_2$-grading. The second grading is the one for which F_V^i ($i = 1$, 2) are purely even, i.e., $F_V^{i+} = F_V^i$. We denote by $\tau_s = \mathrm{Id}$ the involution defined by this $\mathbf{Z}_2$ grading. Then the coefficient of q^n ($n \in \frac{1}{2}\mathbf{Z}$) in (2.13) of $R_1(V)$, $R_2(V)$ (resp. $R_3(V)$, $R_4(V)$) is exactly the $\mathbf{Z}_2$-graded vector sub-bundle of (F_V^1, τ_s), (F_V^1, τ_e) (resp. (F_V^2, τ_e), (F_V^2, τ_s)), on which P acts by multiplication by n.

    We will denote by τ_e (resp. τ_s) the $\mathbf{Z}_2$-grading on $S(TX) \otimes_{n=1}^{\infty} \mathrm{Sym}(TX_n) \otimes F_V^i$ induced by the above $\mathbf{Z}_2$-gradings.

Let h^{V_v} be the metric on V_v induced by the metric h^V on V. In the following, we will identify ΛV_v with $\Lambda \overline{V}_v^*$ by using the Hermitian metric h^{V_v}. By (2.8), as in (1.12), there is a natural isomorphism between $\mathbf{Z}_2$-graded $C(V)$-Clifford modules over F,

$$(3.4) \qquad S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \otimes_{0<v} \Lambda V_v \simeq S(V)_{|F}.$$

By using above notations, on the fixed point set F, we rewrite (2.15), for $p \in \mathbf{N}$,

$$\mathcal{F}_p(X) = \otimes_{0<v} \left(\otimes_{n=1}^{\infty} \mathrm{Sym}(N_{v,n}) \otimes_{\substack{n \in \mathbf{N}, \\ n > pv}} \mathrm{Sym}(\overline{N}_{v,n}) \right) \otimes_{n=1}^{\infty} \mathrm{Sym}(TY_n),$$

$$(3.5) \quad \mathcal{F}_p'(X) = \otimes_{\substack{0<v, n \in \mathbf{N}, \\ 0 \le n \le pv}} \left(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v \right),$$

$$\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}_p'(X).$$

Let $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$. From (2.7), (3.4), we get

$$\mathcal{F}^0(X) = \otimes_{n=1}^{\infty} \mathrm{Sym}\left(\oplus_{0<v}(N_{v,n} \oplus \overline{N}_{v,n}) \right) \otimes_{n=1}^{\infty} \mathrm{Sym}(TY_n)$$
$$\otimes \mathrm{Sym}(\oplus_{0<v} N_{v,0}) \otimes \det(\oplus_{0<v} N_v),$$

$$(3.6) \qquad F_V^1 = \otimes_{n=1}^{\infty} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_{0,n})$$
$$\otimes S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \otimes_{0<v} \Lambda(V_{v,0}),$$

$$F_V^2 = \otimes_{0<n \in \mathbf{Z}+1/2} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_{0,n}).$$

Now, we can reformulate Theorem 2.6 as the following Theorem.

Theorem 3.1 *For each α, h, $p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, for $i = 1,\ 2$, $\tau = \tau_e$ or τ_s, we have the following identity in $K(B)$,*

$$(3.7) \qquad \begin{aligned} &\mathrm{Ind}_\tau(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes F_V^i, m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h) \\ &= \mathrm{Ind}_\tau(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes F_V^i, m + ph + p^2 e, h + 2pe). \end{aligned}$$

Proof: The rest of this section is devoted to the proof of Theorem 3.1. ∎

3.2 Proof of Theorem 3.1

Inspired by [**T**, §7], for $p \in \mathbf{N}^*$, we define the shift operators,

$$(3.8) \qquad \begin{aligned} r_* &: N_{v,n} \to N_{v,n+pv}, & r_* &: \overline{N}_{v,n} \to \overline{N}_{v,n-pv}, \\ r_* &: V_{v,n} \to V_{v,n+pv}, & r_* &: \overline{V}_{v,n} \to \overline{V}_{v,n-pv}. \end{aligned}$$

This means that we change the action of the operator P on $N_{v,n}$ (resp. $\overline{N}_{v,n}$) by $n + pv$ (resp. $n - pv$), etc.

Recall that $L(N)$, $L(V)$ are the complex line bundles over F defined by (2.10). Also recall that $L = L(N)^{-1} \otimes L(V)$ is a trivial complex line bundle on F, and $g \in S^1$ acts on it by multiplication by g^{2e}.

Proposition 3.1 *For $p \in \mathbf{Z}$, $p > 0$, $i = 1,\ 2$, there are natural isomorphisms of vector bundles over F,*

$$(3.9) \qquad \begin{aligned} r_*(\mathcal{F}^{-p}(X)) &\simeq \mathcal{F}^0(X) \otimes L(N)^p, \\ r_*(F_V^i) &\simeq F_V^i \otimes L(V)^{-p}. \end{aligned}$$

Proof: 1) Under the action of the shift operator r_*,

$$
\begin{aligned}
(3.10) \qquad r_*(\mathcal{F}'_p(X)) &= \bigotimes_{\substack{0<v \\ 0\leq n\leq pv}} \left(\mathrm{Sym}(N_{v,-n+pv}) \otimes \det N_v \right) \\
&= \bigotimes_{\substack{0<v \\ 0\leq n\leq pv}} \mathrm{Sym}(N_{v,n}) \otimes_{0<v} \det N_v \otimes L(N)^p,
\end{aligned}
$$

From (3.5), (3.10), we get the first equation of (3.9).

2) For F_V^i ($i = 1,\ 2$), we only need to consider the shift operator on the following elements

$$
\begin{aligned}
(3.11) \qquad F_{V,F}^1 &= \bigotimes_{n=1}^{\infty} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n})) \otimes_{0<v} \Lambda(V_{v,0}), \\
F_{V,F}^2 &= \bigotimes_{n\in\mathbf{N}+1/2} \Lambda(\oplus_{0<v}(V_{v,n} \oplus \overline{V}_{v,n})).
\end{aligned}
$$

We compute easily that

$$
\begin{aligned}
(3.12) \qquad r_*F_{V,F}^1 &= \bigotimes_{n=1}^{\infty} \Lambda(\oplus_{0<v}(V_{v,n+pv} \oplus \overline{V}_{v,n-pv})) \otimes_{0<v} \Lambda(V_{v,pv}), \\
r_*F_{V,F}^2 &= \bigotimes_{n\in\mathbf{N}+1/2} \Lambda(\oplus_{0<v}(V_{v,n+pv} \oplus \overline{V}_{v,n-pv})).
\end{aligned}
$$

The Hermitian metric h^{V_v} on V_v induces a natural isomorphism of complex vector bundles over F,

$$
(3.13) \qquad \Lambda^i \overline{V}_v \simeq \Lambda^{\dim V_v - i} V_v \otimes \det \overline{V}_v.
$$

In fact, let dv_{V_v} be the volume form on $(V_{v,\mathbf{R}}, h^{V_v})$, then we define $\Phi : \Lambda^{\dim V_v - i} V_v \otimes \det \overline{V}_v \to (\Lambda^i V_v)^*$ as follows: for $s_1 \in \Lambda^{\dim V_v - i} V_v \otimes \det \overline{V}_v$, $s_2 \in \Lambda^i V_v$,

$$
\Phi(s_1)(s_2)dv_{V_v} = s_1 \wedge s_2.
$$

Clearly, Φ is an isomorphism of complex vector bundles. By using the Hermitian metric h^{V_v}, we identify $(\Lambda^i V_v)^*$ to $\Lambda^i \overline{V}_v$.

For $n \in \mathbf{N}$, $0 < n \leq pv$, $0 \leq i \leq \dim V_v$, (3.13) induces a natural S^1-equivariant isomorphism of complex vector bundles

$$
\begin{aligned}
(3.14) \qquad \Lambda^i \overline{V}_{v,n-pv} &\simeq \Lambda^{\dim V_v - i} V_{v,-n+pv} \otimes \det \overline{V}_v, \\
\Lambda^i \overline{V}_{v,n-pv-\frac{1}{2}} &\simeq \Lambda^{\dim V_v - i} V_{v,-n+pv+\frac{1}{2}} \otimes \det \overline{V}_v.
\end{aligned}
$$

This means

$$
\begin{aligned}
(3.15) \qquad \bigotimes_{n\in\mathbf{N},0<n\leq pv} \Lambda^{i_n} \overline{V}_{v,n-pv} &\simeq \bigotimes_{n\in\mathbf{N},0<n\leq pv} \left(\Lambda^{\dim V_v - i_n} V_{v,-n+pv} \otimes \det \overline{V}_v \right), \\
\bigotimes_{n\in\mathbf{N},0\leq n<pv} \Lambda^{i_n} \overline{V}_{v,n-pv+\frac{1}{2}} &\simeq \bigotimes_{n\in\mathbf{N},0\leq n<pv} \left(\Lambda^{\dim V_v - i_n} V_{v,-n+pv-\frac{1}{2}} \otimes \det \overline{V}_v \right).
\end{aligned}
$$

From (2.10) and the isomorphisms (3.12) and (3.15) of complex vector bundles over F, one gets the second induced isomorphism in (3.9).

The proof of Proposition 3.1 is complete. ∎

Proposition 3.2 *For $p \in \mathbf{Z}$, $p > 0$, $i = 1$, 2, the bundle isomorphism induced by (3.9),*

(3.16)
$$r_* : S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}^{-p}(X) \otimes F_V^i$$
$$\to S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}^0(X) \otimes F_V^i \otimes L^{-p},$$

verifies the following identities:

(3.17)
$$r_*^{-1} \cdot \mathbf{J}_H \cdot r_* = \mathbf{J}_H,$$
$$r_*^{-1} \cdot P \cdot r_* = P + p\mathbf{J}_H + p^2 e - \tfrac{1}{2}p^2 e(N) - \tfrac{p}{2}d'(N).$$

For the $\mathbf{Z}_2$-gradings, we have

(3.18)
$$\tau_e r_* = r_* \tau_e, \qquad \tau_s r_* = r_* \tau_s.$$

Proof: The first equation of (3.17) is obvious.

To prove the second equation of (3.17) we divide the argument into several steps.

a) From (3.15), on $\bigotimes_{\substack{n\in\mathbf{N}\\0<n\le pv}} \Lambda^{i_n}\overline{V}_{v,n}$, we have

(3.19)
$$r_*^{-1}Pr_* = \sum_{\substack{n\in\mathbf{N}\\0<n\le pv}} (\dim V_v - i_n)(-n + pv)$$
$$= P + p\mathbf{J}_H + \sum_{\substack{n\in\mathbf{N}\\0<n\le pv}} (-n + pv)\dim V_v$$
$$= P + p\mathbf{J}_H + \tfrac{1}{2}(p^2v^2 - pv)\dim V_v.$$

Thus, from (2.9), (3.19), on $\bigotimes_{0<v}\bigotimes_{\substack{n\in\mathbf{N}\\0<n\le pv}} \Lambda^{i_n}\overline{V}_{v,n}$, we have

(3.20)
$$r_*^{-1}Pr_* = P + p\mathbf{J}_H + \frac{1}{2}p^2 e(V) - \frac{1}{2}pd'(V).$$

The operators P and $\mathbf{J}_H$ act on $S\left(V_0^{\mathbf{R}}, \det(\oplus_{0<v}V_v)^{-1}\right)$ by multiplication by

(3.21)
$$0, \quad -\frac{1}{2}\Sigma_{0<v}v\dim V_v = -\frac{1}{2}d'(V).$$

respctively.

When P acts the rest part of F_V^1, we have

(3.22)
$$r_*^{-1}Pr_* = P + p\mathbf{J}_H.$$

From (3.6), (3.20), (3.21) and (3.22), we know that when acting on F_V^1, one has the equality:

(3.23)
$$r_*^{-1}Pr_* = P + p\mathbf{J}_H + \frac{1}{2}p^2 e(V).$$

b) Similar to (3.20), from (3.15), on $\bigotimes_{\substack{n\in\mathbf{N}\\0\le n<pv}} \Lambda^{i_n}\overline{V}_{v,n+\frac{1}{2}}$, we have

(3.24)
$$r_*^{-1}Pr_* = \sum_{\substack{n\in\mathbf{N}\\0\le n<pv}} (\dim V_v - i_n)(-n + pv - \tfrac{1}{2})$$
$$= P + p\mathbf{J}_H + (\dim V_v)\sum_{\substack{n\in\mathbf{N}\\0\le n<pv}} (-n + pv - \tfrac{1}{2}) = P + p\mathbf{J}_H + \tfrac{1}{2}p^2v^2\dim V_v.$$

From (2.9), (3.24), on $\bigotimes_{0<v} \bigotimes_{\substack{n\in\mathbf{N} \\ 0\leq n<pv}} \Lambda^{i_n}\overline{V}_{v,n+\frac{1}{2}}$, we have

$$(3.25) \qquad r_*^{-1}Pr_* = P + p\mathbf{J}_H + \frac{1}{2}p^2 e(V).$$

When P acts on the rest part of F_V^2, one has

$$(3.26) \qquad r_*^{-1}Pr_* = P + p\mathbf{J}_H.$$

From (3.25), (3.26), on F_V^2, we again have (3.23).

c) Note that on $\otimes_{0<v,0\leq n\leq pv} \det N_v$, $\mathbf{J}_H$ acts as $pe(N) + d'(N)$. By (3.5), we know that on $\mathcal{F}^{-p}(X)$,

$$(3.27) \qquad r_*^{-1}Pr_* = P + p\mathbf{J}_H - p(pe(N) + d'(N)).$$

On $S(TY, \det(\oplus_{0<v}N_v)^{-1})$, $\mathbf{J}_H$ acts as $-\frac{1}{2}d'(N)$. So on $S(TY, \det(\oplus_{0<v}N_v)^{-1}) \otimes \mathcal{F}^{-p}(X)$

$$(3.28) \qquad r_*^{-1}Pr_* = P + p\mathbf{J}_H - p^2 e(N) - \frac{1}{2}pd'(N).$$

From (2.12), (3.23), b) and (3.28), we get the second equation of (3.17).

Finally, under our operations, the $\mathbf{Z}_2$-grading τ_s does not change. For the $\mathbf{Z}_2$-grading τ_e, it changes only on $\bigotimes_{0<v} \bigotimes_{\substack{n\in\mathbf{N} \\ 0<n\leq pv}} \Lambda^{i_n}\overline{V}_{v,n}$ of F_V^1 (resp. on $\bigotimes_{0<v} \bigotimes_{\substack{n\in\mathbf{N} \\ 0\leq n<pv}} \Lambda^{i_n}\overline{V}_{v,n+\frac{1}{2}}$ of F_V^2). From (3.15), we know that

$$(3.29) \qquad r_*^{-1}\tau_e r_* = (-1)^{\Sigma_{0<v}pv\dim V_v}\tau_e.$$

As $\frac{1}{2}p_1(V)_{S^1} \in H^*_{S^1}(M,\mathbf{Z})$ is well-defined, from (2.5), (2.11), we get

$$(3.30) \qquad \sum_{0<v} v\dim V_v = 0 \bmod(2),$$

From (3.29), (3.30), we get (3.18).

The proof of Proposition 3.2 is complete. ∎

Proof of Theorem 3.1: From (2.12) and Propositions 3.2, for each α, h, $p \in \mathbf{Z}$, $p > 0$, $m \in \frac{1}{2}\mathbf{Z}$, and for $i = 1$, 2, $\tau = \tau_e$ or τ_s, we have the following indentity in $K(B)$,

$$(3.31) \qquad \begin{aligned} &\mathrm{Ind}_\tau(D^{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes F_V^i, m + \tfrac{1}{2}p^2 e(N) + \tfrac{1}{2}pd'(N), h) \\ &= \mathrm{Ind}_\tau(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes F_V^i \otimes L^{-p}, m + ph + p^2 e, h) \\ &= \mathrm{Ind}_\tau(D^{Y_\alpha} \otimes \mathcal{F}^0(X) \otimes F_V^i, m + ph + p^2 e, h + 2pe). \end{aligned}$$

The proof of Theorem 3.1 is complete. ∎

4   Proof of Theorem 2.7

In this section, we prove Theorem 2.7. Many arguments in this section are inspired by [**T**, §6, 9]. We will construct a family twisted Dirac operator on $M(n_j)$, the fixed point set of the induced $\mathbf{Z}_{n_j}$ action on M. By applying our K-theory version of the equivariant family index theorem to this operator, we derive Theorem 2.7.

This section is organized as follows: In Section 4.1, we construct a family Dirac operator on $M(n_j)$. In Section 4.2, by introducing a shift operator, we will relate both sides of equation (2.26) to the index bundle of the family Dirac operator on $M(n_j)$. In Section 4.3, we prove Theorem 2.7.

In this section, we make the same assumptions and use the same notations as in Sections 2 and 3.

4.1   The Spin^c Dirac operator on $M(n_j)$

Let $\pi : M \to B$ be a fibration of compact manifolds with fiber X and $\dim_{\mathbf{R}} X = 2l$. We assume that S^1 acts fiberwise on M, and TX has an S^1-invariant spin structure. Let V be a real vector bundle over M carrying an S^1-invariant spin structure and $\dim_{\mathbf{R}} V = 2k$.

Let $F = \{F_\alpha\}$ be the fixed point set of the S^1 action on M. Then $\pi : F \to B$ is a fibration with compact fiber Y. For $n \in \mathbf{N}, \ n > 0$, let $\mathbf{Z}_n \subset S^1$ denote the cyclic subgroup of order n.

For $n_j \in \mathbf{N}$ with $n_j > 0$, let $M(n_j)$ be the fixed point set of the induced $\mathbf{Z}_{n_j}$-action on M. Then $\pi : M(n_j) \to B$ is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \to M(n_j)$ be the normal bundle of $M(n_j)$ in M. Then we have the following $\mathbf{Z}_{n_j}$-equivariant decomposition of $N(n_j) \otimes_{\mathbf{R}} \mathbf{C}$ over $M(n_j)$,

$$(4.1) \qquad N(n_j) \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{0 < v < n_j} N(n_j)_v.$$

Here $N(n_j)_v$ is the complex vector bundles over $M(n_j)$ with $g \in \mathbf{Z}_{n_j}$ acting by g^v on it. Complex conjugation provides a $\mathbf{C}$ anti-linear isomorphism between $N(n_j)_v$ and $\overline{N(n_j)_{n_j-v}}$. If n_j is even, this produces a real structure on $N(n_j)_{\frac{n_j}{2}}$, so this bundle is the complexification of a real vector bundle $N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ on $M(n_j)$. Thus, $N(n_j)$ is isomorphic, as a real vector bundle, to

$$(4.2) \qquad N(n_j) \simeq \bigoplus_{0 < v < n_j/2} N(n_j)_v \oplus N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}.$$

Similarly, we have the following $\mathbf{Z}_{n_j}$-equivariant decomposition of $V \otimes_{\mathbf{R}} \mathbf{C}$,

$$(4.3) \qquad V \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{0 \leq v < n_j} V(n_j)_v.$$

Here $V(n_j)_v$ is the complex vector bundle over $M(n_j)$ with $g \in \mathbf{Z}_{n_j}$ acting by g^v on it. For $v \neq 0$, complex conjugation provides a $\mathbf{C}$ anti-linear isomorphism between $V(n_j)_v$ and $\overline{V(n_j)_{n_j-v}}$. If n_j is even, this produces a real structure on $V(n_j)_{\frac{n_j}{2}}$, so this bundle is

the complexification of a real vector bundle $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ over $M(n_j)$. Complex conjugation also provides a real structure on $V(n_j)_0$ such that $V(n_j)_0 = V(n_j)^{\mathbf{R}}_0 \otimes_{\mathbf{R}} \mathbf{C}$. Thus, over $M(n_j)$, V is isomorphic, as a real vector bundle, to

$$(4.4) \qquad V|_{M(n_j)} \simeq V(n_j)^{\mathbf{R}}_0 \bigoplus_{0<v<n_j/2} V(n_j)_v \oplus V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}.$$

In (4.2), (4.4), the last term is understood to be zero when n_j is odd.

It is essential for us to know that the vector bundles $TX(n_j)$ and $V(n_j)^{\mathbf{R}}_0$ are orientable. For this we have the following Lemma which was proved in [**E**] and [**BT**, Lemma 10.1].

Lemma 4.1 *Let W be a real, spin vector bundle over a manifold M. We assume that $\mathbf{Z}_n$ $(n \in \mathbf{N}^*)$ acts on M, and that the $\mathbf{Z}_n$ action lifts on W and preserves the spin structure of W. Let $M(n)$ be the fixed point set of the $\mathbf{Z}_n$ action on M. Let W_0 be the subbundle of W over $M(n)$ on which the generator of $\mathbf{Z}_n$ acts trivially. Then W_0 is orientable.*

By Lemma 4.1, $TX(n_j)$ and $V(n_j)^{\mathbf{R}}_0$ are orientable over $M(n_j)$. Thus $N(n_j)$ is orientable over $M(n_j)$. By (4.2), (4.4), $N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ and $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ are also orientable over $M(n_j)$. In the following, we fix the orientations of $N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$, $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ over $M(n_j)$. Then $TX(n_j)$ and $V(n_j)^{\mathbf{R}}_0$ are naturally oriented by (4.2), (4.4) and the orientations of TX, $V, N(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}, V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$.

Let us denote by

$$(4.5) \qquad r(n_j) = \frac{1}{2}(1 + (-1)^{n_j}).$$

Lemma 4.2 *Assume that (2.6) holds. Let*

$$(4.6) \qquad L(n_j) = \bigotimes_{0<v<n_j/2} \left(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \right)^{(r(n_j)+1)v}$$

be the line bundle over $M(n_j)$. Then:

  i) $L(n_j)$ has an n_j^{th} root over $M(n_j)$.
  ii) Let

$$(4.7) \qquad \begin{aligned} L_1 &= \bigotimes_{0<v<n_j/2} \left(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \right) \otimes L(n_j)^{r(n_j)/n_j}, \\ L_2 &= \bigotimes_{0<v<n_j/2} \left(\det(N(n_j)_v) \right) \otimes L(n_j)^{r(n_j)/n_j}. \end{aligned}$$

Let $U_1 = TX(n_j) \oplus V(n_j)^{\mathbf{R}}_0$ and $U_2 = TX(n_j) \oplus V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$. Then $TX(n_j)$, $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ and $V(n_j)^{\mathbf{R}}_0$ are of even dimensions. Furthermore, U_1 (resp. U_2) has a Spin^c structure defined by L_1 (resp. L_2).

Proof: By [**BT**, Lemma 9.4], $TX(n_j)$, $V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}$ and $V(n_j)^{\mathbf{R}}_0$ are of even dimensions. From the proof of [**BT**, Lemmas 11.3 and 11.4], we get the rest part of Lemma 4.2. ∎

Lemma 4.2 is very important. It allows us, as we are going to see, to apply the constructions and results in Section 1.3 to the fibration $M(n_j) \to B$, which is the main concern of this Section.

For $p_j \in \mathbf{N}$, $p_j < n_j$, $(p_j, n_j) = 1$, $\beta_j = \frac{p_j}{n_j}$, let us write

$$(4.8) \quad \mathcal{F}(\beta_j) = \otimes_{0<n\in\mathbf{Z}}\mathrm{Sym}(TX(n_j)_n) \bigotimes_{0<v<n_j/2} \mathrm{Sym}\Big(\bigoplus_{0<n\in\mathbf{Z}+p_jv/n_j} N(n_j)_{v,n}$$
$$\bigoplus_{0<n\in\mathbf{Z}-p_jv/n_j} \overline{N(n_j)}_{v,n} \Big) \otimes_{0<n\in\mathbf{Z}+\frac{1}{2}} \mathrm{Sym}(N(n_j)_{\frac{n_j}{2},n}),$$

$$F^1_V(\beta_j) = \Lambda\Big(\oplus_{0<n\in\mathbf{Z}} V(n_j)_{0,n} \bigoplus_{0<v<n_j/2} \Big(\oplus_{0<n\in\mathbf{Z}+p_jv/n_j} V(n_j)_{v,n}$$
$$\oplus_{0<n\in\mathbf{Z}-p_jv/n_j} \overline{V(n_j)}_{v,n} \Big) \oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V(n_j)_{\frac{n_j}{2},n} \Big),$$

$$F^2_V(\beta_j) = \Lambda\Big(\oplus_{0<n\in\mathbf{Z}} V(n_j)_{\frac{n_j}{2},n} \bigoplus_{0<v<n_j/2} \Big(\oplus_{0<n\in\mathbf{Z}+p_jv/n_j+\frac{1}{2}} V(n_j)_{v,n}$$
$$\oplus_{0<n\in\mathbf{Z}-p_jv/n_j+\frac{1}{2}} \overline{V(n_j)}_{v,n} \Big) \oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V(n_j)_{0,n} \Big).$$

We denote by $D^{X(n_j)}$ the S^1-equivariant Spin^c-Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ along the fiber $X(n_j)$ defined as in Section 1.3. We denote by $D^{X(n_j)} \otimes \mathcal{F}(\beta_j) \otimes F^i_V(\beta_j)$ $(i = 1,\ 2)$ the corresponding twisted Spin^c Dirac operator on $S(U_i, L_i) \otimes \mathcal{F}(\beta_j) \otimes F^i_V(\beta_j)$ along the fiber $X(n_j)$.

Remark 4.1. In fact, to define an S^1-action on $L(n_j)^{r(n_j)/n_j}$, one must replace the S^1-action by its n_j-fold action. Here by abusing notation, we still say an S^1-action without causing any confusion.

In the rest of this subsection, we will reinterpret all of the above objects when we restrict ourselves to F, the fixed point set of the S^1 action. We will use the notation of Sections 1.3, 2.

Let $N_{F/M(n_j)}$ be the normal bundle of F in $M(n_j)$. Then by (2.7),

$$(4.9) \quad \begin{aligned} N_{F/M(n_j)} &= \bigoplus_{0<v:v\in n_j\mathbf{Z}} N_v, \\ TX(n_j) \otimes_{\mathbf{R}} \mathbf{C} &= TY \otimes_{\mathbf{R}} \mathbf{C} \oplus_{0<v,v\in n_j\mathbf{Z}} (N_v \oplus \overline{N}_v). \end{aligned}$$

By (2.7) and (4.1), the restriction of $N(n_j)_v$ $(1 \le v \le n_j/2)$ to F is given by

$$(4.10) \quad N(n_j)_v = \bigoplus_{0<v':v'=v \ \mathrm{mod}(n_j)} N_{v'} \bigoplus_{0<v':v'=-v \ \mathrm{mod}(n_j)} \overline{N}_{v'}.$$

By (2.8) and (4.4), the restriction of $V(n_j)_v$ $(1 \le v \le n_j/2)$ to F is given by

$$(4.11) \quad V(n_j)_v = \bigoplus_{0<v':v'=v \ \mathrm{mod}(n_j)} V_{v'} \bigoplus_{0<v':v'=-v \ \mathrm{mod}(n_j)} \overline{V}_{v'}.$$

and for $v = 0$,

$$(4.12) \quad V(n_j)_0 = V^{\mathbf{R}}_0 \otimes_{\mathbf{R}} \mathbf{C} \bigoplus_{0<v,v=0 \ \mathrm{mod}(n_j)} (V_v \oplus \overline{V}_v).$$

From (4.9)-(4.12), we have the following identifications of real vector bundles over F,

(4.13)
$$N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}} = \bigoplus_{0<v,v=\frac{n_j}{2} \mod(n_j)} N_v,$$
$$TX(n_j) = TY \bigoplus_{0<v,v=0 \mod(n_j)} N_v,$$
$$V(n_j)_0^{\mathbf{R}} = V_0^{\mathbf{R}} \bigoplus_{0<v,v=0 \mod(n_j)} V_v,$$
$$V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}} = \bigoplus_{0<v,v=\frac{n_j}{2} \mod(n_j)} V_v.$$

We denote by $V_0 = V_0^{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $V_0^{\mathbf{R}}$ over F. As $(p_j, n_j) = 1$, we know that, for $v \in \mathbf{Z}$, $p_j v/n_j \in \mathbf{Z}$ iff $v/n_j \in \mathbf{Z}$. Also, $p_j v/n_j \in \mathbf{Z} + \frac{1}{2}$ iff $v/n_j \in \mathbf{Z} + \frac{1}{2}$. From (4.9)-(4.12), we then get

(4.14)
$$\mathcal{F}(\beta_j) = \otimes_{0<n\in\mathbf{Z}}\mathrm{Sym}(TY_n) \bigotimes_{0<v,v=0,\frac{n_j}{2} \mod(n_j)} \bigotimes_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} \mathrm{Sym}(N_{v,n} \oplus \overline{N}_{v,n})$$
$$\bigotimes_{0<v'<n_j/2} \mathrm{Sym}\left(\oplus_{v=v' \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} N_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{N}_{v,n} \right) \right.$$
$$\left. \oplus_{v=-v' \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} N_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{N}_{v,n} \right) \right),$$

$$F_V^1(\beta_j) = \Lambda\left[\oplus_{0<n\in\mathbf{Z}} V_{0,n} \bigoplus_{0<v,v=0,\frac{n_j}{2} \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{V}_{v,n} \right) \right.$$
$$\left. \oplus_{0<v'<n_j/2} \left(\oplus_{v=v',-v' \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}} \overline{V}_{v,n} \right) \right) \right],$$
$$F_V^2(\beta_j) = \Lambda\left[\oplus_{0<n\in\mathbf{Z}+\frac{1}{2}} V_{0,n} \oplus_{0<v,v=0,\frac{n_j}{2} \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}+\frac{1}{2}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}+\frac{1}{2}} \overline{V}_{v,n} \right) \right.$$
$$\left. \oplus_{0<v'<n_j/2} \left(\oplus_{v=v',-v' \mod(n_j)} \left(\oplus_{0<n\in\mathbf{Z}+\frac{p_j v}{n_j}+\frac{1}{2}} V_{v,n} \oplus_{0<n\in\mathbf{Z}-\frac{p_j v}{n_j}+\frac{1}{2}} \overline{V}_{v,n} \right) \right) \right].$$

Now we want to compare the spinor bundles over F. From (4.6), (4.7), (4.10) and (4.11), we find that over F we have

(4.15)
$$L(n_j)^{\frac{r(n_j)}{n_j}} = \bigotimes_{0<v'<n_j/2} \left(\otimes_{v=v' \mod(n_j)} (\det N_v \otimes \det \overline{V}_v)^{2v'} \right.$$
$$\left. \otimes_{v=-v' \mod(n_j)} (\det N_v \otimes \det \overline{V}_v)^{-2v'} \right)^{r(n_j)/n_j},$$
$$L_1 = \bigotimes_{0<v'<n_j/2} \left(\otimes_{v=v' \mod(n_j)} (\det N_v \otimes \det \overline{V}_v) \right.$$
$$\left. \otimes_{v=-v' \mod(n_j)} (\det N_v \otimes \det \overline{V}_v)^{-1} \right) \otimes L(n_j)^{r(n_j)/n_j},$$
$$L_2 = \bigotimes_{0<v'<n_j/2} \left(\otimes_{v=v' \mod(n_j)} \det N_v \right.$$
$$\left. \otimes_{v=-v' \mod(n_j)} (\det N_v)^{-1} \right) \otimes L(n_j)^{r(n_j)/n_j}.$$

From (4.13), over F, we have

(4.16)
$$TX(n_j) \oplus V(n_j)_0^{\mathbf{R}} = TY \oplus V_0^{\mathbf{R}} \oplus_{0<v,v=0 \mod(n_j)} (N_v \oplus V_v),$$
$$TX(n_j) \oplus V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}} = TY \oplus_{0<v,v=0 \mod(n_j)} N_v \oplus_{0<v,v=\frac{n_j}{2} \mod(n_j)} V_v.$$

Recall that the Spin^c vector bundles U_1 and U_2 have been defined in Lemma 4.2. Let us write

(4.17)

$$S(U_1, L_1)' = S\Big(TY \oplus V_0^{\mathbf{R}}, L_1 \bigotimes_{\substack{0<v, \\ v=0 \ \mathrm{mod}(n_j)}} (\det N_v \otimes \det V_v)^{-1}\Big) \bigotimes_{\substack{0<v, \\ v=0 \ \mathrm{mod}(n_j)}} \Lambda V_v,$$

$$S(U_2, L_2)' = S\Big(TY, L_2 \bigotimes_{\substack{0<v, \\ v=0 \ \mathrm{mod}(n_j)}} (\det N_v)^{-1} \bigotimes_{\substack{0<v, \\ v=\frac{n_j}{2} \ \mathrm{mod}(n_j)}} (\det V_v)^{-1}\Big) \bigotimes_{\substack{0<v, \\ v=\frac{n_j}{2} \ \mathrm{mod}(n_j)}} \Lambda V_v.$$

Then from (1.47), (1.49) and (4.17), for $i = 1,\ 2$, we have the following isomorphism of Clifford modules over F,

$$(4.18) \qquad S(U_i, L_i) \simeq S(U_i, L_i)' \otimes \Lambda(\oplus_{0<v, v=0 \ \mathrm{mod}(n_j)} N_v).$$

We define the $\mathbf{Z}_2$ gradings on $S(U_i, L_i)'$ $(i = 1,\ 2)$ as that induced by the $\mathbf{Z}_2$-gradings on $S(U_i, L_i)$ $(i = 1,\ 2)$ and on $\Lambda(\oplus_{0<v, v=0 \ \mathrm{mod}(n_j)} N_v)$ such that the isomorphism (4.18) preserves these $\mathbf{Z}_2$-gradings.

We define formally the following complex line bundles over F,

$$L_1' = \Big[L_1^{-1} \otimes_{\substack{0<v, \\ v=0 \ \mathrm{mod}(n_j)}} (\det N_v \otimes \det V_v) \otimes_{0<v} (\det N_v \otimes \det V_v)^{-1}\Big]^{1/2},$$

$$L_2' = \Big[L_2^{-1} \otimes_{\substack{0<v, \\ v=0 \ \mathrm{mod}(n_j)}} \det N_v \otimes_{\substack{0<v, \\ v=n_j/2 \ \mathrm{mod}(n_j)}} \det V_v \otimes_{0<v} (\det N_v)^{-1}\Big]^{1/2}.$$

From (1.11), (1.47), Lemma 4.2 and the assumption that V is spin, one verifies easily that $c_1(L_i'^2) = 0 \ \mathrm{mod}(2)$ for $i = 1,\ 2$. Thus L_1', L_2' are well defined complex line bundles over F. For later use, we also write down the following expressions of L_i' $(i = 1,\ 2)$ which can be deduced from (4.15):

$$(4.19) \quad \begin{aligned} L_1' &= \Big[L(n_j)^{-1/n_j} \otimes_{0<v, v=\frac{n_j}{2} \ \mathrm{mod}(n_j)} (\det N_v \otimes \det \overline{V}_v)\Big]^{\frac{r(n_j)}{2}} \\ &\quad \otimes_{0<v, 0<v\leq \frac{n_j}{2} \ \mathrm{mod}(n_j)} (\det N_v)^{-1} \otimes_{0<v, \frac{n_j}{2}<v<n_j \ \mathrm{mod}(n_j)} (\det V_v)^{-1}, \\ L_2' &= \Big[L(n_j)^{-1/n_j} \otimes_{0<v, v=\frac{n_j}{2} \ \mathrm{mod}(n_j)} (\det N_v \otimes \det V_v)\Big]^{\frac{r(n_j)}{2}} \\ &\quad \otimes_{0<v, 0<v\leq \frac{n_j}{2} \mathrm{mod}(n_j)} (\det N_v)^{-1}. \end{aligned}$$

From (4.15), (4.17) and the definition of L_i' $(i = 1,\ 2)$, we get the following identifications of Clifford modules over F,

$$(4.20) \quad \begin{aligned} S(U_1, L_1)' \otimes L_1' &= S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \\ &\quad \otimes \Lambda(\oplus_{0<v, v=0 \ \mathrm{mod}(n_j)} V_v), \\ S(U_2, L_2)' \otimes L_2' &= S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \Lambda(\oplus_{0<v, v=\frac{n_j}{2} \ \mathrm{mod}(n_j)} V_v). \end{aligned}$$

To compare the $\mathbf{Z}_2$-gradings in (4.20), we will compare explicitly the orientations. Recall that, if (W, h^W) is a real Euclidian vector space of dimension $2m$, and J is a complex structure on W which preserves h^W. Let $\{e_i, Je_i\}_{i=1}^m$ be an orthonormal basis of (W, h^W). Then W is canonically oriented, and its orientation is defined by the canonical Riemannian volume form

$$(e_1 \wedge Je_1) \wedge \cdots (e_m \wedge Je_m) = dv_W.$$

Let dv_{TX}, dv_V be the corresponding Riemannian volume forms on (TX, h^{TX}) and (V, h^V) which define the orientations of TX, V over M. Let dv_{N_v} and $dv_{\overline{N}_v}$ (resp. dv_{V_v}, $dv_{\overline{V}_v}$) $(0 < v)$ be the canonical Riemannian volume forms on $N_{v,\mathbf{R}}$ and $\overline{N}_{v,\mathbf{R}} = N_{v,\mathbf{R}}$ (resp. $V_{v,\mathbf{R}}$, $\overline{V}_{v,\mathbf{R}} = V_{v,\mathbf{R}}$). Then through the identifications (1.12) and (3.4), the orientations of TY and $V_0^{\mathbf{R}}$ over F are defined by the volume forms dv_{TY} and $dv_{V_0^{\mathbf{R}}}$ respectively such that

$$(4.21) \qquad \begin{aligned} dv_{TX} &= dv_{TY} \otimes_{0<v} dv_{N_v}, \\ dv_V &= dv_{V_0^{\mathbf{R}}} \otimes_{0<v} dv_{V_v}. \end{aligned}$$

By (4.2), (4.4), (4.10) and (4.11), the orientations of $TX(n_j)$ and $V(n_j)_0^{\mathbf{R}}$, when restricted to F, are given by

$$(4.22)$$

$$dv_{TX}|_F = dv_{TX(n_j)} \bigotimes_{0<v'<n_j/2} \left(\bigotimes_{v=v' \bmod(n_j)} dv_{N_v} \bigotimes_{v=-v' \bmod(n_j)} dv_{\overline{N}_v} \right) \otimes dv_{N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}},$$

$$dv_V|_F = dv_{V(n_j)_0^{\mathbf{R}}} \bigotimes_{0<v'<n_j/2} \left(\bigotimes_{v=v' \bmod(n_j)} dv_{V_v} \bigotimes_{v=-v' \bmod(n_j)} dv_{\overline{V}_v} \right) \otimes dv_{V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}}.$$

respectively.

Clearly, we have

$$(4.23) \qquad \begin{aligned} dv_{N_v} &= (-1)^{\dim N_v} dv_{\overline{N}_v}, \\ dv_{V_v} &= (-1)^{\dim V_v} dv_{\overline{V}_v}. \end{aligned}$$

From (4.21), (4.22) and (4.23), we get

$$(4.24) \qquad \begin{aligned} dv_{TX(n_j)} &= (-1)^{\Delta(n_j, N)} dv_{TY} \bigotimes_{0<v, v=0 \bmod(n_j)} dv_{N_v}, \\ dv_{V(n_j)_0^{\mathbf{R}}} &= (-1)^{\Delta(n_j, V)} dv_{V_0^{\mathbf{R}}} \bigotimes_{0<v, v=0 \bmod(n_j)} dv_{V_v}, \end{aligned}$$

where

$$(4.25) \qquad \begin{aligned} \Delta(n_j, N) &= \sum_{\frac{n_j}{2}<v'<n_j} \sum_{0<v=v' \bmod(n_j)} \dim N_v + o(N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}), \\ \Delta(n_j, V) &= \sum_{\frac{n_j}{2}<v'<n_j} \sum_{0<v=v' \bmod(n_j)} \dim V_v + o(V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}), \end{aligned}$$

with $o(N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}) = 0$ or 1 (resp. $o(V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}) = 0$ or 1), the value depends on whether the given orientation on $N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$ (resp. $V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}$) agrees or disagrees with the complex orientation of $\oplus_{v=\frac{n_j}{2} \bmod(n_j)} N_v$ (resp. $\oplus_{v=\frac{n_j}{2} \bmod(n_j)} V_v$).

From (4.13), (4.18), (4.23) and (4.24), we see that, for the $\mathbf{Z}_2$-gradings induced by τ_s, the difference of the $\mathbf{Z}_2$-gradings of (4.20) is $(-1)^{\Delta(n_j, N)}$; for the $\mathbf{Z}_2$-gradings induced by τ_e, the difference of the $\mathbf{Z}_2$-gradings of the first (resp. second) equation of (4.20) is $(-1)^{\Delta(n_j, N) + \Delta(n_j, V)}$ (resp. $(-1)^{\Delta(n_j, N) + o(V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}})}$).

4.2 The Shift operators

Let $p \in \mathbf{N}^*$ be fixed. For any $1 \leq j \leq J_0$, inspired by [**T**, §9], we define the following shift operators r_{j*}:

(4.26)
$$r_{j*} : N_{v,n} \to N_{v,n+(p-1)v+p_j v/n_j}, \quad r_{j*} : \overline{N}_{v,n} \to \overline{N}_{v,n-(p-1)v-p_j v/n_j},$$
$$r_{j*} : V_{v,n} \to V_{v,n+(p-1)v+p_j v/n_j}, \quad r_{j*} : \overline{V}_{v,n} \to \overline{V}_{v,n-(p-1)v-p_j v/n_j}.$$

This means that we change the action of the operator P on $N_{v,n}$ by a multiplication by $n + (p-1)v + p_j v/n_j$, etc.

If E is a combination of the above bundles, we denote by $r_{j*}E$ the bundle on which the action of P is changed in the above way.

From (2.22), (3.5), we get

(4.27)
$$\mathcal{F}_{p,j}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}'_{p-1}(X) \bigotimes_{(v,n)\in\cup_{i=1}^j I_i^p} \left(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v \right)$$
$$\bigotimes_{(v,n)\in\overline{I}_j^p} \mathrm{Sym}(\overline{N}_{v,n}).$$

Recal that the vector bundles F_V^i $(i = 1, \ 2)$ have been defined in (3.6).

Proposition 4.1 *There are natural isomorphisms of vector bundles over F,*

(4.28)
$$r_{j*}\mathcal{F}_{p,j-1}(X) \simeq \mathcal{F}(\beta_j) \bigotimes_{0<v,v=0 \ \mathrm{mod}(n_j)} \mathrm{Sym}(\overline{N}_{v,0})$$
$$\otimes_{0<v}(\det N_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1} \bigotimes_{0<v,v=0 \ \mathrm{mod}(n_j)}(\det N_v)^{-1},$$
$$r_{j*}\mathcal{F}_{p,j}(X) \simeq \mathcal{F}(\beta_j) \bigotimes_{0<v,v=0 \ \mathrm{mod}(n_j)} \mathrm{Sym}(N_{v,0}) \otimes_{0<v} (\det N_v)^{[\frac{p_j v}{n_j}]+(p-1)v+1},$$
$$r_{j*}F_V^1 \simeq S(V_0^{\mathbf{R}}, \otimes_{0<v}(\det V_v)^{-1}) \otimes F_V^1(\beta_j) \bigotimes_{0<v,v=0 \ \mathrm{mod}(n_j)} \Lambda(V_{v,0})$$
$$\otimes_{0<v}(\det \overline{V}_v)^{[\frac{p_j v}{n_j}]+(p-1)v},$$
$$r_{j*}F_V^2 \simeq F_V^2(\beta_j) \bigotimes_{0<v,v=\frac{n_j}{2} \ \mathrm{mod}(n_j)} \Lambda(V_{v,0}) \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_j v}{n_j}+\frac{1}{2}]+(p-1)v}.$$

Proof: The proof is similar to that of Proposition 3.1. We divide it into several steps.
1) From (3.5), (4.26), we get

(4.29) $\quad r_{j*}\mathcal{F}'_{p-1}(X) = \bigotimes_{\substack{0<v,n\in\mathbf{N} \\ 0\leq n\leq(p-1)v}} \mathrm{Sym}(N_{v,-n+(p-1)v+\frac{p_j v}{n_j}}) \otimes_{0<v} (\det N_v)^{(p-1)v+1}.$

Note that by (2.20), for $v \in J = \{v \in \mathbf{N} |$ There exists α such that $N_v \neq 0$ on $F_\alpha\}$, there are no integer in $]\frac{p_{j-1}v}{n_{j-1}}, \frac{p_j v}{n_j}[$. So for $v \in J$ and $i_0 = j-1, \ j$, the elements $(v,n) \in \cup_{i=1}^{i_0} I_i^p$ are $(v, (p-1)v+1), \cdots, (v, (p-1)v+[\frac{p_{i_0} v}{n_{i_0}}])$. Furthermore,

(4.30)
$$[\frac{p_{j-1}v}{n_{j-1}}] = [\frac{p_j v}{n_j}] - 1 \quad \text{if} \quad v = 0 \quad \mathrm{mod}(n_j),$$

$$[\frac{p_{j-1}v}{n_{j-1}}] = [\frac{p_j v}{n_j}] \quad \text{if} \quad v \neq 0 \quad \mathrm{mod}(n_j).$$

From (2.21), (4.26) and (4.30), we have

$$r_{j*}\Big(\bigotimes_{(v,n)\in\cup_{i=1}^{j-1} I_i^p} \big(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v \big) \Big)$$

$$= \bigotimes_{(v,n)\in\cup_{i=1}^{j-1} I_i^p} \mathrm{Sym}(N_{v,-n+(p-1)v+\frac{p_jv}{n_j}}) \otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]}$$

(4.31)
$$\otimes_{\substack{0<v,\\ v=0 \mod(n_j)}} (\det N_v)^{-1},$$

$$r_{j*}\Big(\bigotimes_{(v,n)\in\cup_{i=1}^{j} I_i^p} \big(\mathrm{Sym}(N_{v,-n}) \otimes \det N_v \big) \Big)$$

$$= \bigotimes_{(v,n)\in\cup_{i=1}^{j} I_i^p} \mathrm{Sym}(N_{v,-n+(p-1)v+\frac{p_jv}{n_j}}) \otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]}.$$

From (3.5), (4.14), (4.27), (4.29), (4.31), one easily gets the first two equations in (4.28).

2) For $0 < n \le (p-1)v + \frac{p_jv}{n_j}$, $n \in \mathbf{Z}$, $0 \le i \le \dim V_v$, (3.13) induces a natural S^1-equivariant isomorphism of complex vector bundles over F,

(4.32)
$$\Lambda^i \overline{V}_{v,n-(p-1)v-\frac{p_jv}{n_j}} \simeq \Lambda^{\dim V_v - i} V_{v,-n+(p-1)v+\frac{p_jv}{n_j}} \otimes \det \overline{V}_v.$$

From (3.6) and (4.32), as in (3.9), we obtain the third equality of (4.28).

3) For $0 < n \le (p-1)v + \frac{p_jv}{n_j} + \frac{1}{2}$, $n \in \mathbf{Z}$, $0 \le i \le \dim V_v$, (3.13) induces a natural S^1-equivariant isomorphism of complex vector bundles over F,

(4.33)
$$\Lambda^i \overline{V}_{v,n-(p-1)v-\frac{p_jv}{n_j}-\frac{1}{2}} \simeq \Lambda^{\dim V_v - i} V_{v,-n+(p-1)v+\frac{p_jv}{n_j}+\frac{1}{2}} \otimes \det \overline{V}_v.$$

From (3.6), (4.33), as in (3.9), we get the last equality of (4.28).

The proof of Proposition 4.1 is complete. ∎

Lemma 4.3 *Let us introduce the following two line bundles,*

(4.34)
$$L(\beta_j)_1 = L_1' \otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_jv}{n_j}]+(p-1)v}$$
$$\otimes_{0<v,v=0 \mod(n_j)} (\det N_v)^{-1},$$

$$L(\beta_j)_2 = L_2' \otimes_{0<v} (\det N_v)^{[\frac{p_jv}{n_j}]+(p-1)v+1} \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_jv}{n_j}+\frac{1}{2}]+(p-1)v}$$
$$\otimes_{0<v,v=0 \mod(n_j)} (\det N_v)^{-1}.$$

Then $L(\beta_j)_1$, $L(\beta_j)_2$ can be extended naturally to S^1-equivariant complex line bundles which we will still denote by $L(\beta_j)_1$, $L(\beta_j)_2$ respectively over $M(n_j)$.

Proof: We divide the argument into several steps.

1) Write

(4.35)
$$[\frac{p_jv}{n_j}] = \frac{p_jv}{n_j} - \frac{\omega(v)}{n_j}.$$

Note that for $v = \frac{n_j}{2} \mod(n_j)$, $\frac{\omega(v)}{n_j} = \frac{1}{2}$. From (2.10), (4.19) and (4.35), we get, formally,

(4.36)
$$L(\beta_j)_1 = L^{-(p-1)-p_j/n_j} \otimes_{0<v,v\ne\frac{n_j}{2} \mod(n_j)} (\det N_v \otimes \det \overline{V}_v)^{-\frac{\omega(v)}{n_j}}$$
$$\otimes_{0<v,\frac{n_j}{2}<v<n_j \mod(n_j)} (\det N_v \otimes \det \overline{V}_v) \otimes L(n_j)^{\frac{-r(n_j)}{2n_j}}.$$

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Note that, if $v = v' \mod(n_j)$, then $\omega(v) = \omega(v')$. Also, for $0 < v' < n_j$, $\omega(n_j - v') = n_j - \omega(v')$. From (4.6), (4.10), (4.11) and (4.36), formally, we get

(4.37)
$$L(\beta_j)_1 = L^{-(p-1)-p_j/n_j}$$
$$\otimes \Big[\bigotimes_{0<v<\frac{n_j}{2}} \Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \Big)^{-\omega(v)-r(n_j)v} \Big]^{1/n_j}.$$

We introduce the following line bundle over $M(n_j)$,

(4.38)
$$L^\omega(\beta_j) = \bigotimes_{0<v<\frac{n_j}{2}} \Big(\det(N(n_j)_v) \otimes \det(\overline{V(n_j)}_v) \Big)^{-\omega(v)-r(n_j)v}.$$

Note that $\omega(v) = p_j v \mod(n_j)$. If n_j is odd, then $r(n_j) = 0$, Lemma 4.2 implies that $L^\omega(\beta_j)^{1/n_j}$ is well defined over $M(n_j)$. If n_j is even, then p_j is odd, and $r(n_j) = 1$, Lemma 4.2 again implies $L^\omega(\beta_j)^{1/n_j}$ is well defined over $M(n_j)$. So $L(\beta_j)_1$ extends naturally over $M(n_j)$.

2) From (2.10), (4.19), (4.34), we get

$$L(\beta_j)_2 = L^{-(p-1)} \otimes_{0<v} (\det N_v)^{[\frac{p_j v}{n_j}]} \otimes_{0<v} (\det \overline{V}_v)^{[\frac{p_j v}{n_j}+\frac{1}{2}]} \otimes_{\substack{0<v, \\ \frac{n_j}{2}<v<n_j \mod(n_j)}} \det N_v$$

(4.39)
$$\otimes \Big[\otimes_{\substack{0<v, \\ v=\frac{n_j}{2} \mod(n_j)}} (\det N_v \otimes \det V_v) \Big]^{r(n_j)/2} \otimes L(n_j)^{-r(n_j)/2n_j}.$$

By using the same argument as in 1), one deduces that

(4.40)
$$\otimes_{0<v}(\det N_v)^{-\omega(v)/n_j} \bigotimes_{\substack{0<v, \\ \frac{n_j}{2}<v<n_j \mod(n_j)}} \det N_v \otimes \Big[\bigotimes_{v=\frac{n_j}{2} \mod(n_j)} \det N_v \Big]^{r(n_j)/2}$$
$$= \bigotimes_{0<v<\frac{n_j}{2}} (\det(N(n_j)_v))^{-\omega(v)/n_j}.$$

Note that, if there exists $m \in [0, p_j[$ with $m \in \mathbf{N}$, such that $m \le p_j v/n_j < (m+\frac{1}{2})$, then

(4.41)
$$[\frac{p_j v}{n_j} + \frac{1}{2}] = p_j v/n_j - \omega(v)/n_j.$$

While if there exists $m \in]0, p_j]$, $m \in \mathbf{N}$ such that $(m - \frac{1}{2}) \le p_j v/n_j < m$, then

(4.42)
$$[\frac{p_j v}{n_j} + \frac{1}{2}] = p_j v/n_j + (n_j - \omega(v))/n_j.$$

From (4.11), we have the following formal identity,

$$\bigotimes_{\substack{0<v'<n_j, m\in\mathbf{N} \\ m<p_j v'/n_j<m+\frac{1}{2}}} \bigotimes_{v=v' \mod(n_j)} (\det \overline{V}_v)^{-\frac{\omega(v)}{n_j}} \bigotimes_{\substack{0<v'<n_j, m\in\mathbf{N} \\ m-\frac{1}{2}<p_j v'/n_j<m}} \bigotimes_{v=v' \mod(n_j)} (\det \overline{V}_v)^{\frac{(n_j-\omega(v))}{n_j}}$$

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$$(4.43) \qquad = \bigotimes_{\substack{0<v'<n_j/2, m\in\mathbf{N} \\ m<p_jv'/n_j<m+\frac{1}{2}}} \left(\bigotimes_{v=v' \bmod(n_j)} (\det \overline{V}_v)^{-\frac{\omega(v')}{n_j}} \bigotimes_{v=-v' \bmod(n_j)} (\det \overline{V}_v)^{\frac{\omega(v')}{n_j}} \right)$$

$$\bigotimes_{\substack{0<v'<n_j/2, m\in\mathbf{N} \\ m-\frac{1}{2}<p_jv'/n_j<m}} \left(\bigotimes_{v=-v' \bmod(n_j)} (\det \overline{V}_v)^{-\frac{(n_j-\omega(v'))}{n_j}} \bigotimes_{v=v' \bmod(n_j)} (\det \overline{V}_v)^{\frac{(n_j-\omega(v'))}{n_j}} \right)$$

$$= \bigotimes_{0<v'<n_j/2} (\det(\overline{V(n_j)}_{v'}))^{-\frac{\omega(v')}{n_j}} \bigotimes_{1\le m\le p_j/2} \bigotimes_{m-\frac{1}{2}<p_jv'/n_j<m} \det(\overline{V(n_j)}_{v'}).$$

Recall that for $v \in \mathbf{Z}$, $p_jv/n_j \in \mathbf{Z}$ iff $v/n_j \in \mathbf{Z}$, also $p_jv/n_j \in \mathbf{Z}+\frac{1}{2}$ iff $v/n_j \in \mathbf{Z}+\frac{1}{2}$. From (2.10), (4.39)-(4.43), we get

$$(4.44) \qquad L(\beta_j)_2 = L^{-(p-1)-\frac{p_j}{n_j}} \otimes L^\omega(\beta_j)^{\frac{1}{n_j}} \bigotimes_{1\le m\le p_j/2} \bigotimes_{m-\frac{1}{2}<p_jv'/n_j<m} \det(\overline{V(n_j)}_{v'}).$$

The proof of Lemma 4.3 is complete. ∎

Let us denote by

$$\varepsilon_1 = \tfrac{1}{2}\sum_{0<v}(\dim N_v - \dim V_v)\Big[([\tfrac{p_jv}{n_j}]+(p-1)v)([\tfrac{p_jv}{n_j}]+(p-1)v+1)$$
$$-(\tfrac{p_jv}{n_j}+(p-1)v)\Big(2\Big([\tfrac{p_jv}{n_j}]+(p-1)v\Big)+1\Big)\Big],$$

(4.45)
$$\varepsilon_2 = \tfrac{1}{2}\sum_{0<v}(\dim N_v)\Big[([\tfrac{p_jv}{n_j}]+(p-1)v)([\tfrac{p_jv}{n_j}]+(p-1)v+1)$$
$$-(\tfrac{p_jv}{n_j}+(p-1)v)\Big(2([\tfrac{p_jv}{n_j}]+(p-1)v)+1\Big)\Big]$$
$$-\tfrac{1}{2}\sum_{0<v}(\dim V_v)\Big[([\tfrac{p_jv}{n_j}+\tfrac{1}{2}]+(p-1)v)^2$$
$$-2(\tfrac{p_jv}{n_j}+(p-1)v)([\tfrac{p_jv}{n_j}+\tfrac{1}{2}]+(p-1)v)\Big].$$

Then ε_1, ε_2 are locally constant functions on F.

Recall that, if E is a S^1-equivariant vector bundle over M, then the weight of the S^1 action on $\Gamma(F,E)$ is given by the action $\mathbf{J}_H$ (cf. Section 3.1).

Proposition 4.2 *For $i = 1$, 2, the isomorphisms induced by (4.20) and (4.28),*

$$r_{i1}: S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}_{p,j-1}(X) \otimes F_V^i \to$$
$$S(U_i, L_i)' \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \otimes L(\beta_j)_i \otimes_{\substack{0<v, \\ v=0\bmod(n_j)}} \mathrm{Sym}(\overline{N}_{v,0}),$$

(4.46)
$$r_{i2}: S(TY, \otimes_{0<v}(\det N_v)^{-1}) \otimes \mathcal{F}_{p,j}(X) \otimes F_V^i \to$$
$$S(U_i, L_i)' \otimes \mathcal{F}(\beta_j) \otimes F_V^i(\beta_j) \otimes L(\beta_j)_i \otimes_{\substack{0<v, \\ v=0 \bmod(n_j)}} (\mathrm{Sym}(N_{v,0}) \otimes \det N_v),$$

have the following properties :
1) for $i = 1$, 2, $\gamma = 1$, 2, one has

$$(4.47) \qquad \begin{aligned} r_{i\gamma}^{-1}\mathbf{J}_H r_{i\gamma} &= \mathbf{J}_H, \\ r_{i\gamma}^{-1}P r_{i\gamma} &= P + (\tfrac{p_j}{n_j}+(p-1))\mathbf{J}_H + \varepsilon_{i\gamma}, \end{aligned}$$

where

$$\text{(4.48)} \qquad \begin{aligned} \varepsilon_{i1} &= \varepsilon_i - e(p, \beta_{j-1}, N), \\ \varepsilon_{i2} &= \varepsilon_i - e(p, \beta_j, N). \end{aligned}$$

2) Recall that $o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}})$ was defined in (4.25). Denote by

$$\text{(4.49)} \qquad \begin{aligned} \mu_1 &= -\sum_{0<v}[\tfrac{p_j v}{n_j}]\dim V_v + \Delta(n_j, N) + \Delta(n_j, V) \quad \mathrm{mod}(2), \\ \mu_2 &= -\sum_{0<v}[\tfrac{p_j v}{n_j} + \tfrac{1}{2}]\dim V_v + \Delta(n_j, N) + o(V(n_j)^{\mathbf{R}}_{\frac{n_j}{2}}) \quad \mathrm{mod}(2), \\ \mu_3 &= \Delta(n_j, N) \quad \mathrm{mod}(2). \end{aligned}$$

Then, for $i = 1,\ 2,\ \gamma = 1,\ 2$, one has

$$\text{(4.50)} \qquad \begin{aligned} r_{i\gamma}^{-1}\tau_e r_{i\gamma} &= (-1)^{\mu_i}\tau_e, \\ r_{i\gamma}^{-1}\tau_s r_{i\gamma} &= (-1)^{\mu_3}\tau_s. \end{aligned}$$

Proof: The first equality of (4.47) is trivial.
From (2.24) and (4.30), one has

$$\text{(4.51)} \quad e(p, \beta_j, N) = e(p, \beta_{j-1}, N) + \sum_{0<v, v=0 \ \mathrm{mod}(n_j)} \left((p-1)v + \frac{p_j v}{n_j}\right)\dim N_v.$$

For $i = 1,\ 2$, denote by $\varepsilon_i(V)$ the contribution of $\dim V$ to ε_i. From (4.32), when acting on $\bigotimes_{\substack{0<v,n\in\mathbf{N} \\ 0<n\leq(p-1)v+\frac{p_j v}{n_j}}} \Lambda^{in}\overline{V}_{v,n}$, as in (3.20), we get that

$$\text{(4.52)} \qquad \begin{aligned} r_{j*}^{-1}P r_{j*} &= P + ((p-1) + \tfrac{p_j}{n_j})\mathbf{J}_H \\ &\quad + \sum_{\substack{0<v,n\in\mathbf{N}, \\ 0<n\leq(p-1)v+\frac{p_j v}{n_j}}} (\dim V_v)(-n + (p-1)v + \tfrac{p_j v}{n_j}) \\ &= P + ((p-1) + \tfrac{p_j}{n_j})\mathbf{J}_H + \varepsilon_1(V) - \tfrac{1}{2}((p-1) + \tfrac{p_j}{n_j})d'(V). \end{aligned}$$

Similarly, by (4.33), when acting on $\bigotimes_{\substack{0<v,n\in\mathbf{N} \\ 0<n\leq(p-1)v+\frac{p_j v}{n_j}+\frac{1}{2}}} \Lambda^{in}\overline{V}_{v,n-\frac{1}{2}}$, as in (3.24), we get that

$$\text{(4.53)} \qquad \begin{aligned} r_{j*}^{-1}P r_{j*} &= P + ((p-1) + \tfrac{p_j}{n_j})\mathbf{J}_H \\ &\quad + \sum_{\substack{0<v,n\in\mathbf{N}, \\ 0<n\leq(p-1)v+\frac{p_j v}{n_j}+\frac{1}{2}}} (\dim V_v)(-n + (p-1)v + \tfrac{p_j v}{n_j} + \tfrac{1}{2}) \\ &= P + ((p-1) + \tfrac{p_j}{n_j})\mathbf{J}_H + \varepsilon_2(V). \end{aligned}$$

From (3.21), (4.29)-(4.31), (4.51), (4.52), (4.53), and by proceeding as in the proof of Proposition 3.2, one deduces easily the second equation of (4.47).

3) From (4.32), on $\bigotimes_{\substack{0<v,n\in\mathbf{N} \\ 0<n\leq(p-1)v+\frac{p_j v}{n_j}}} \Lambda^{in}\overline{V}_{v,n}$, we have

$$\text{(4.54)} \qquad r_{j*}^{-1}\tau_e r_{j*} = (-1)^{-\Sigma_{0<v}([\frac{p_j v}{n_j}]+(p-1)v)\dim V_v}\tau_e.$$

From (4.33), on $\bigotimes_{\substack{0<v,n\in\mathbf{N} \\ 0<n\leq(p-1)v+\frac{p_jv}{n_j}+\frac{1}{2}}} \Lambda^{in}\overline{V}_{v,n-\frac{1}{2}}$, we have

$$(4.55) \qquad r_{j*}^{-1}\tau_e r_{j*} = (-1)^{-\Sigma_{0<v}([\frac{p_jv}{n_j}+\frac{1}{2}]+(p-1)v)\dim V_v}\tau_e.$$

As we don't change the $\mathbf{Z}_2$-grading in the rest part of (4.46), from (3.30), (4.17), (4.54), (4.55) and the discussion following (4.25), we get (4.50).

The proof of Proposition 4.2 is complete. ∎

The following Lemmas 4.4 and 4.6 were essentially proved in [$\mathbf{T}$, Lemmas 9.6, 9.7].

Lemma 4.4 *For each connected component M' of $M(n_j)$, ε_1, ε_2 are independent on the connected component of F in M'.*

Proof: From (2.12), (4.11), (4.13), (4.35) and (4.45), we get

$$\varepsilon_1 = \frac{1}{2}\sum_{0\leq v'<n_j}\sum_{0<v,v=v'\ \mathrm{mod}(n_j)}(\dim N_v - \dim V_v)\left[-(\frac{p_jv}{n_j}+(p-1)v)^2 - \frac{\omega(v')(n_j-\omega(v'))}{n_j^2}\right]$$

$$(4.56) \qquad = (p-1+\frac{p_j}{n_j})^2 e - \frac{1}{16}\left(\dim_{\mathbf{R}} N(n_j)_{\frac{n_j}{2}}^{\mathbf{R}} - \dim_{\mathbf{R}} V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}\right)$$

$$-\frac{1}{2}\sum_{0<v'<n_j/2}\left(\dim N(n_j)_{v'} - \dim V(n_j)_{v'}\right)\frac{\omega(v')(n_j-\omega(v'))}{n_j^2}.$$

By combining (4.11), (4.13), (4.16), (4.41), (4.42) and (4.45), as in (4.56), we get

$$\varepsilon_2 = \varepsilon_1 - \frac{1}{2}\sum_{0\leq m<p_j}\sum_{m<p_jv'/n_j<m+\frac{1}{2}}(\dim V(n_j)_{v'})\frac{\omega(v')}{n_j}$$

$$(4.57) \qquad -\frac{1}{2}\sum_{0<m\leq p_j}\sum_{m-\frac{1}{2}<p_jv'/n_j<m}(\dim V(n_j)_{v'})\frac{(n_j-\omega(v'))}{n_j} - \frac{1}{8}\dim_{\mathbf{R}} V(n_j)_{\frac{n_j}{2}}^{\mathbf{R}}.$$

The proof of Lemma 4.4 is complete. ∎

The following Lemma was proved in [$\mathbf{BT}$, Lemma 9.3].

Lemma 4.5 *Let M be a smooth manifold on which S^1 acts. Let M' be a connected component of $M(n_j)$, the fixed point set of $\mathbf{Z}_{n_j}$ subgroup of S^1 on M. Let F be the fixed point set of S^1-action on M. Let $V \to M$ be a real, oriented, even dimensional vector bundle to which the S^1-action on M lifts. Assume that V is spin over M. Let $p_j \in]0, n_j[$, $p_j \in \mathbf{N}$ and $(p_j, n_j) = 1$, then*

$$(4.58) \qquad \sum_{0<v}(\dim V_v)[\frac{p_jv}{n_j}] + \Delta(n_j, V) \quad \mathrm{mod}(2)$$

is independent on the connected components of F in M'.

Recall that the number $d'(\beta_j, N)$ has been defined in (2.24).

Lemma 4.6 *For each connected component M' of $M(n_j)$, $d'(\beta_j, N) + \mu_i \mod(2)$ ($i = 1,\ 2,\ 3$) is independent on the connected component of F in M'.*

Proof : The assertions for the cases $i = 1,\ 3$ follow immediately from Lemma 4.5 by replacing V with TX.

From (4.41) and (4.42), we have

$$o(V(n_j)^{\mathbf{R}}_{n_j/2}) + \sum_{0<v}(\dim V_v)[\frac{p_j v}{n_j} + \frac{1}{2}] = \sum_{0<v}(\dim V_v)[\frac{p_j v}{n_j}] + o(V(n_j)^{\mathbf{R}}_{n_j/2})$$

$$(4.59) + \sum_{\substack{0<m\le p_j}} \sum_{\substack{0<v'<n_j \\ m-\frac{1}{2}<p_j v'/n_j<m}} \sum_{\substack{0<v, \\ v=v' \mod(n_j)}} \dim V_v + \sum_{\substack{0<v, \\ v=\frac{n_j}{2} \mod(n_j)}} \dim V_v.$$

By (4.13), the last term in (4.59), $\dim_{\mathbf{R}} V(n_j)^{\mathbf{R}}_{n_j/2}$, is a locally constant function on $M(n_j)$.

By (4.11), the third term on the right side of (4.59) is equal to

$$(4.60) \sum_{n_j/2<v'<n_j} \sum_{0<v,v=v' \mod(n_j)} \dim V_v + \sum_{0<m\le p_j} \sum_{\substack{0<v'<\frac{n_j}{2} \\ m-\frac{1}{2}<p_j v'/n_j<m}} \dim V(n_j)_{v'} \quad \mod(2)$$

The last term of (4.60) is a locally constant function on $M(n_j)$.

By (4.25), the first term in (4.60) is $o(V(n_j)^{\mathbf{R}}_{n_j/2}) + \Delta(n_j, V) \mod(2)$. On the other hand, by Lemma 4.5, we know that $\Sigma_{0<v}(\dim V_v)[\frac{p_j v}{n_j}] + \Delta(n_j, V) \mod(2)$ is independent on the component of F in M'.

By the above discussion, the left side of (4.59) is $\mod(2)$ independent on the component of F in M'.

The proof of Lemma 4.6 is complete. ∎

4.3 Proof of Theorem 2.7

From (2.24), (4.10), (4.13) and (4.30), we have

$$\sum_{0<v}\dim N_v = \sum_{0<v<\frac{n_j}{2}} \dim N(n_j)_v + \frac{1}{2}\dim_{\mathbf{R}} N(n_j)^{\mathbf{R}}_{n_j/2} + \sum_{0<v,v=0 \mod(n_j)} \dim N_v,$$

$$(4.61) \quad d'(\beta_j, N) = d'(\beta_{j-1}, N) + \sum_{0<v,v=0 \mod(n_j)} \dim N_v.$$

By Lemma 4.6, and (4.61), $d'(\beta_{j-1}, N) + \sum_{0<v} \dim N_v + \mu_i \mod(2)$ is a constant function on each connected component of $M(n_j)$.

From Lemma 4.3, one knows that the Dirac operator $D^{X(n_j)} \otimes F(\beta_j) \otimes F^i_V(\beta_j) \otimes L(\beta_j)_i$ ($i = 1,\ 2$) is well-defined on $M(n_j)$. Thus, by using Proposition 4.2, Lemma 4.4, (4.9), (4.18) and (4.61), for $i = 1,\ 2$, $h \in \mathbf{Z}$, $m \in \frac{1}{2}\mathbf{Z}$, $\tau = \tau_e$ or τ_s, and by applying separately

both the first and the second equations of Theorem 1.2 to each connected component of $M(n_j)$, we get the following identity in $K(B)$,

(4.62)
$$\sum_\alpha (-1)^{d'(\beta_{j-1},N)+\sum_{0<v}\dim N_v} \mathrm{Ind}_\tau (D^{Y_\alpha} \otimes \mathcal{F}_{p,j-1}(X) \otimes F_V^i, m + e(p, \beta_{j-1}, N), h)$$
$$= \sum_\beta (-1)^{d'(\beta_{j-1},N)+\sum_{0<v}\dim N_v+\mu} \mathrm{Ind}_\tau (D^{X(n_j)} \otimes F(\beta_j) \otimes F_V^i(\beta_j) \otimes L(\beta_j)_i,$$
$$m + \varepsilon_i + (\tfrac{p_j}{n_j} + (p-1))h, h)$$
$$= \sum_\alpha (-1)^{d'(\beta_j,N)+\sum_{0<v}\dim N_v} \mathrm{Ind}_\tau (D^{Y_\alpha} \otimes \mathcal{F}_{p,j}(X) \otimes F_V^i, m + e(p, \beta_j, N), h).$$

Here $\sum_\beta$ means the sum over all the connected components of $M(n_j)$. In (4.62), if $\tau = \tau_s$, then $\mu = \mu_3$; if $\tau = \tau_e$, then $\mu = \mu_i$.

The proof of Theorem 2.7 is complete. ∎

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